



বিদ্যাসাগর বিশ্ববিদ্যালয়  
VIDYASAGAR UNIVERSITY  
Question Paper

**B.Sc. Honours Examinations 2022**

(Under CBCS Pattern)

**Semester - IV**

**Subject : MATHEMATICS**

**Paper : C 10 - T**

**Ring Theory and Linear Algebra - I**

**Full Marks : 60**

**Time : 3 Hours**

*Candidates are required to give their answers in their own words as far as practicable.  
The figures in the margin indicate full marks.*

1. Answer any **five** questions : 2×5=10
- (a) Let  $X$  be any set and  $R$  be the power set of  $X$ . Does  $(R, +, \cdot)$  form a ring where  $A + B = A \cup B$  and  $A \cdot B = A \cap B$  for all  $A, B \in R$ .
- (b) Extend  $S = \{(1,1,0), (1,1,1)\}$  to a basis of the vector space  $\mathbb{R}^3$  over  $\mathbb{R}$ .
- (c) Find the total number of units in the ring  $M_2(\mathbb{Z}_3)$ , with usual notations.
- (d) Let  $V$  be a vector space over the field  $F$  and  $W_1, W_2$  be two subspaces of  $V$ . Is  $W_1 \cup W_2$  a subspace of  $V$ ?
- (e) In the ring  $M_2(\mathbb{Z})$  of all  $2 \times 2$  matrices over  $\mathbb{Z}$ , check whether the set  $\left\{ \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} \mid a, b \in \mathbb{Z} \right\}$  forms an ideal or not.

P.T.O.

- (f) Determine the linear operator  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  over  $\mathbb{R}$  such that  $T$  sends the vectors  $(1,0,0)$ ,  $(0,1,0)$ ,  $(0,0,1)$  to  $(0,1,0)$ ,  $(0,0,1)$ ,  $(1,0,0)$ , respectively.
- (g) Show that the mapping  $f : \mathbb{Z}\sqrt{2} \rightarrow M_2(\mathbb{R})$  (where  $M_2(\mathbb{R})$  denotes the ring of all  $2 \times 2$  real matrices) defined by  $f(a + b\sqrt{2}) = \begin{pmatrix} a & 2b \\ b & a \end{pmatrix}$  is a homomorphism of rings.
- (h) Give an example of a linear operator  $T$  on a vector space  $V$  such that  $\ker T = \text{Im } T$ .

2. Answer any **four** questions :

5×4=20

- (a) Let  $I$  denote the set of all polynomials in  $\mathbb{Z}[x]$  with constant term of the form  $4k$  ( $k \in \mathbb{Z}$ ). Show that  $I$  is an ideal of  $\mathbb{Z}[x]$ . Is it a prime ideal? Is it a maximal ideal? Give proper justification in support of your answer. 2+2+1=5
- (b) Let  $V$  be a vector space over the field  $F$  and  $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$  be a basis for  $V$ . Let  $\beta \in V$  be a non-null vector such that  $\beta = c_1\alpha_1 + c_2\alpha_2 + \dots + c_n\alpha_n$  for some  $c_1, c_2, \dots, c_n \in F$  where  $c_k \neq 0$  for some  $1 \leq k \leq n$ . Then show that  $\{\alpha_1, \alpha_2, \dots, \alpha_{k-1}, \beta, \alpha_{k+1}, \dots, \alpha_n\}$  is also a basis for  $V$ . 5
- (c) Let  $(F, +, \cdot)$  be a field and  $u (\neq 0) \in F$ . Define multiplication  $\times$  in  $F$  by  $a \times b = a \cdot u \cdot b$  for  $a, b \in F$ . Prove that  $(F, +, \times)$  is a field.
- (d) Find a non-identity linear transformation  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  such that the  $T(W) = W$  where  $W = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$ .
- (e) Let  $R$  be a ring and  $S$  be a non-empty subset of  $R$ . Show that  $M = \{a \in R \mid ax = 0 \text{ for all } x \in S\}$  is a left ideal of  $R$ . Give an example to show that  $M$  need not be always an ideal of  $R$ . 2+3=5
- (f) Let  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be a linear transformation defined by  $T(a, b, c) = (a + b, 2c - a)$  for all  $(a, b, c) \in \mathbb{R}^3$ . Find the matrix representation of  $T$  relative to the pair of bases  $B = \{(1, 0, -1), (1, 1, 1), (1, 0, 0)\}$  and  $B' = \{(0, 1), (1, 0)\}$ .

P.T.O.

3. Answer any **three** questions :

10×3=30

(a) (i) Let  $R$  be the ring of all continuous function from  $\mathbb{R}$  to  $\mathbb{R}$ . Show that  $A = \{f \in R \mid f(0) = 0\}$  is a maximal ideal of  $R$ .

(ii) Check whether the rings  $\mathbb{Z}[i]$  and  $\mathbb{Z}[\sqrt{2}]$  is isomorphic.

(iii) Let  $V$  be a real vector space with three subspaces  $P, Q, R$  satisfying  $V = P \cup Q \cup R$ . Prove that at least one of  $P, Q, R$  must be  $V$  itself.

3+3+4=10

(b) (i) Let  $I$  and  $J$  be two ideals of a ring  $R$ . Find the smallest ideal of  $R$  containing both  $I$  and  $J$ .

(ii) Give an example to show that quotient ring of an integral domain is not always an integral domain.

(iii) The matrix of a linear operator  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  with respect to the ordered

basis  $B = \{(-1, 1, 1), (1, -1, 1), (1, 1, -1)\}$  is  $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 3 \\ 3 & 3 & 1 \end{pmatrix}$ . Find the matrix of  $T$  with respect to the ordered basis  $B_1 = \{(0, 1, 1), (1, 0, 1), (1, 1, 0)\}$ .

3+2+5=10

(c) (i) Let  $S = \{\frac{a}{b} \in \mathbb{Q} \mid \gcd(a, b) = 1 \text{ and } 3 \text{ does not divide } b\}$ . Show that  $S$  is a ring under usual addition and multiplication of rational numbers. Also prove that  $M = \{\frac{a}{b} \in S \mid 3 \text{ divides } a\}$  is an ideal of  $S$  and the quotient ring  $S/M$  is a field.

(ii) Let  $V, W$  be two finite dimensional vector spaces over the same field  $F, T : V \rightarrow W$  be a linear transformation. Then prove that following are equivalent : (A)  $T$  carries each linearly independent subset of  $V$  to a linearly independent subset of  $W$ . (B)  $\ker T = \{\theta\}$ .

(2+2+2)+4=10

(d) (i) Prove that the ring  $Z_n$  is a principal ideal ring.

P.T.O.

- (ii) Find all non-trivial ring homomorphisms from the ring  $\mathbb{Z}_{12}$  to the ring  $\mathbb{Z}_{28}$ .
- (iii) Let  $U, V, W$  be three finite dimensional vector spaces over the field  $F, T:V \rightarrow W$  be a linear transformation and  $S:W \rightarrow U$  be an isomorphism. Then prove that (A)  $\dim \ker T = \dim \ker ST$  and (B)  $\dim \operatorname{Im} T = \dim \operatorname{Im} ST$ .  
3+3+4=10
- (e) (i) In a commutative ring  $R$  with unity, then show that an ideal  $P$  is a prime ideal if and only if the quotient ring  $\frac{R}{P}$  is an integral domain.
- (ii) Give an example of an infinite ring with finite characteristic.
- (iii) Let  $U$  and  $W$  be two subspaces of a vector space  $V$  over the field  $F$ . Prove that  $U \cup W$  is a subspace of  $V$  if and only if either  $U \subseteq W$  or  $W \subseteq U$ .  
4+2+4=10

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