2022

5th Semester Examination

MATHEMATICS (General)

Paper : DSE 1A/2A/3A-T

[CBCS]

Full Marks : 60 Time : Three Hours

The figures in the margin indicate full marks. Candidates are required to give their answers in their own words as far as practicable.

[Complex Analysis]

1. Answer any *ten* questions : 2×10=20

   (a) Show that $f(z)=|z|^2$ is continuous for all $z \in C$.

   (b) Define analytic function in complex plane. Give an example.

   (c) If $z_1=-1+i$ and $z_2=2-3i$, evaluate $\operatorname{Im}\left(\frac{\overline{z_1}}{iz_2}\right)$.

   (d) Prove that $f(z)=\overline{z}$ is nowhere differentiable on C.

   (e) State Cauchy-Riemann equation.

   (f) Define harmonic function.

   (g) Show that $\lim_{z\to 0}\frac{\overline{z}}{z}$ does not exist.

(h) Define extended complex plane.

(i) Show that if $\underset{z \to z_0}{Lt} f(z)$ exists, then it must be unique.

(j) Prove that the function

$$u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$$

satisfies Laplace's equation.

(k) State the Liouville's theorem.

(l) Give examples of absolute and uniform convergence of power series.

(m) Show that the function $f(z) = \frac{z - \sin z}{z^3}$ has a removable singularity at $z = 0$.

(n) Expand $f(z) = \frac{z-1}{z+1}$ as a Taylor's series about $z = 0$.

(o) Find the domain of convergence of power series

$$\Sigma\left(1 - \frac{1}{n}\right)^{n^2} . z^n .$$

2. Answer any ***four*** questions : 5×4=20

(a) If the real part of the complex no $\frac{z-i}{z-1}$ is zero,

then show that the complex no z lies on the circle with centre $\frac{1+i}{2}$ and radius $\frac{1}{\sqrt{2}}$.

(b) Consider the function f defined by

$$f(z) = \begin{cases} 0, \text{ when } z = 0 \\ \frac{x^3 - y^3}{x^2 + y^2} + i\frac{\left(x^3 + y^3\right)}{x^2 + y^2}, \text{ when } z \neq 0 \end{cases}$$

Show that the function f satisfies the Cauchy-Riemann equation at the origin, but f is not differentiable at $z = 0$.

(c) Show that an analytic function with constant modulus is constant.

(d) Show that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \equiv 4\frac{\partial^2}{\partial z \partial \bar{z}}$.

(e) Construct an analytic function $f(z)$ whose real part is $e^x \cos y$.

(f) State and proof the fundamental theorem of integral calculus in the complex plane.

3. Answer any ***two*** questions : 10×2=20

(a) (i) Find the residue of $F(z) = \frac{\cot z . \coth z}{z^3}$ at $z = 0$.

P.T.O.

(ii) Evaluate $\int_0^\infty \frac{dx}{x^6+1}$.

(b) (i) State and prove Laurent's theorem.

(ii) Prove that an absolute convergent series is convergent.

(c) (i) Let $f(z)=u(x,y)+iv(x,y)$, $z=x+iy$ and $z_0=x_0+iy$. Let the function f be defined in a domain except possible at the point z_0 in D. Then $\underset{z\to z_0}{Lt} f(z)=w_0=u_0+iv_0$ iff $\lim_{z\to z_0} u(x,y)=u_0$ and $\underset{z\to z_0}{Lt} v(x,y)=v_0$.

(ii) Let $f(z)=\begin{cases}\frac{|z|}{\mathrm{Re}(z)} & \text{if } \mathrm{Re}(z)\neq 0\\ 0 & \text{if } \mathrm{Re}(z)=0.\end{cases}$

Show that f is not continuous at $z=0$.

(d) (i) Expand $f(z)=\frac{1}{(z+1)(z+3)}$ in a Laurent series for $|z|<1$.

(ii) Show that the function $f(z)=\sqrt{|xy|}$ is not regular at the origin, although Cauchy-Riemann equations are satisfied at the point.

OR

[Matrices]

1. Answer any *ten* questions : 2×10=20

(a) For what value of k, the following system of equations $x+ky+3z=0$, $3x+ky-2z=0$, $2x+3y-4z=0$ possesses a non-trivial solution.

(b) Let V be a real vector space with $\{\alpha,\beta,\gamma\}$ as a basis. Prove that the set $\{\alpha+\beta+\gamma, \beta+\gamma, \gamma\}$ is also a basis.

(c) Let $S=\{(x,y,z)\in R^3 : x+y+z=0\}$, prove that S is a subspace of R^3.

(d) Prove that an elementary row operation of the first kind does not alter the row rank of a matrix.

(e) Prove that a matrix and its transpose have the same eigenvalues.

(f) Determine the value for which the system of equations

$$x+y+z=6$$

$$2x+y+3z=13$$

$$5x+2y+az=33,$$

has only one solution.

(g) Find the eigen values of the matrix $A=\begin{pmatrix}1 & -i\\ i & 1\end{pmatrix}$.

(h) Find the all real λ for which the rank of the following matrix A is 2 :

$$A=\begin{bmatrix}1 & 2 & 3 & 1\\ 2 & 5 & 3 & \lambda\\ 1 & 1 & 6 & \lambda+1\end{bmatrix}$$

(i) Verify Cayley-Hamilton theorem for the matrix :

$$\begin{bmatrix}1 & 2\\ -1 & 1\end{bmatrix}$$

(j) If A is an $n\times n$ matrix and there exists a unique matrix B such that $AB=I_n$, then prove that $BA=I_n$.

(k) In R^2, if $\alpha=(3,1)$, $\beta=(2,-1)$, then determine $L\{\alpha,\beta\}$.

(l) Show that the matrix $A=\begin{pmatrix}1 & 0\\ 3 & 1\end{pmatrix}$ is not diagonalizable.

(m) Express (5, 2, 1) as a linear combination of (1, 4, 0), (2, 2, 1) and (3, 0, 1).

(n) Find a basis and the dimension of the subspace W of R^3, where $W=\{(x,y,z)\in R^3; x+y+z=0\}$.

(o) Let P is a real orthogonal matrix with det $P=-1$. Prove that -1 is an eigen value of P.

2. Answer any ***four*** questions : 5×4=20

(a) If $A=\begin{pmatrix}4 & 2 & 2\\ 2 & 4 & 2\\ 2 & 2 & 4\end{pmatrix}$,

show that $A^2-10A+16I_3=0$ and then find A^{-1}. 2+3

(b) Prove that a set of vectors containing null vector is linearly dependent. 5

(c) Find all real values of α for which the rank of the matrix $\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & 2 & -1 & \alpha\\ 5 & 7 & 1 & \alpha^2\end{bmatrix}$ is 2. 5

(d) If $A=\begin{pmatrix}0 & 1 & -1\\ -2 & 2 & 0\\ -2 & 1 & 1\end{pmatrix}$ is a 3×3 real matrix, then show that 0 is an eigen value of the matrix A and also find the eigen vector corresponding to the eigen value 0. 2+3

P.T.O.

(e) Diagonalise the symmetric matrix $A = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix}$. 5

(f) Prove that the row rank and column rank of any matrix are identical. 5

3. Answer any *two* questions : 10×2=20

(a) (i) Define the eigenbasis for a square matrix. 2

(ii) For the matrix $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$, find eigenbasis in R^3, if exists. 8

(b) (i) Determine the conditions for which the system $x + y + z = 1$, $x + 2y - z = b$, $5x + 7y + az = b^2$ admits of (i) only one solution, (ii) no solution, (iii) many solutions.

(ii) If λ be an eigen value of a non-singular matrix A, then prove that λ^{-1} is an eigen value of A^{-1}. 7+3

(c) (i) Define diagonalizable matrix. 2

(ii) If possible diagonalize the matrix

$\begin{bmatrix} -1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix}$. Also find the diagonalizing matrix, if exists. 8

(d) (i) Find the inverse of the matrix A by elementary row operations, where

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 4 & 4 \\ 3 & 3 & 7 \end{pmatrix}.$$

(ii) Prove that the set $S = \{(1, 0, 1), (0, 1, 1), (1, 1, 0)\}$ is a basis of R^3. 6+4

OR

[Linear Algebra]

1. Answer any ***ten*** questions : $2\times10=20$

(a) For a vector space over the field F, prove that $cv=\theta \Rightarrow$ either $c=0$ or $v=\theta$.

(b) Define subspace of a vector space, V over the field F.

(c) If S and T are two non-empty finite subsets of a vector space $V(F)$ and $S\subset T$, then prove that $L(S)\subset L(T)$.

(d) Examine that the subset {(1, 2, 0), (3, −1, 1), (4, 1, 1)} is linearly dependent or not.

(e) Prove that union of two subspaces of a vector space $V(F)$ may not a subspace of $V(F)$.

(f) Find the basis of the subspace W of R^3, when $W=\{(x,y,z)\,|\,x+2y-3z=0\}$.

(g) Let $V(F)$ be a vector space of dimension n. Prove that any linearly independent set of n vectors of V is a basis of V.

(h) Define quotient subspace.

(i) Define linear transformation from a vector space V to W over the field F.

(j) Prove that a linear transformation $T:V\to W$ is injective iff $\ker(T)=\{\theta\}$.

(k) Prove that dim V is an odd integer if dim ker T = dim Im T for a linear transformation $T:V\to W$.

(l) Find the matrix of the linear transformation $T:R^3\to R^2$ is defined by $T(x,y,z)=(3x-2y+z,\ x-3y-2z)$, $(x,y,z)\in R^3$ with relative to the order basis {(0, 1, 0), (1, 0, 0), (0, 0, 1)} of R^3 and {(0, 1), (1, 0)}.

(m) Let V and W be the vector space over the field of F. If a linear mapping $T:V\to W$ be invertible then prove that the inverse mapping $T^{-1}:W\to V$ is linear.

(n) A mapping $T:R^3\to R^3$ is defined by $T(x,y,z)=(yz,zx,xy)$ where $(x,y,z)\in R^3$. Examine whether T is linear or not.

(o) Prove that a linear mapping $f:R^2\to R^2$ defined by $f(x,y)=(x\cos\alpha-y\sin\alpha,\ x\sin\alpha+y\cos\alpha)$ is an isomorphism. (α is a given constant)

2. Answer any ***four*** questions : $5\times4=20$

(a) If $\{\alpha_1,\alpha_2,\ldots,\alpha_n\}$ is a basis of a vector space $V(F)$ and a non-zero vector ξ of V is expressed as $\xi=c_1\alpha_1+c_2\alpha_2+\cdots+c_n\alpha_n$, $c_i\in F$, then

P.T.O.

$c_k \neq 0$ prove that $\{\alpha_1, \alpha_2, ..., \alpha_{k-1}, \xi, \alpha_{k+1}, ..., \alpha_n\}$ is a new basis of V.

(b) Let $V(F)$ be a vector space of a finite dimension and W is a subspace of V. Then prove that $\dim W \leq \dim V$.

(c) Let W be a subspace of a vector space $V(F)$. Prove that for $\alpha, \beta \in V$ the cosets $\alpha + W$ and $\beta + W$ are equivalent iff $\alpha - \beta \in W$.

(d) Find the linear transformation $T: R^3 \to R^2$ which maps the basis vectors (1, 0, 0), (0, 1, 0) and (0, 0, 1) of R^3 to the vectors (1, 1), (2, 3) and (3, 2) of R^2 respectively.

(e) Let V be a finite dimensional vector space over the field F and $(\alpha_1, \alpha_2, ..., \alpha_n)$ be an ordered basis of V. A linear mapping $T: V \to V$ as such that $T(\alpha_i) = \alpha_{i+1}$ for $i = 1, 2, ..., n-1$ and $T(\alpha_n) = \alpha_1$. Prove that $T^n = I$, I being the identity mapping on V.

(f) Find the linear mapping $T: R^3 \to R^3$ if $T(1, 0, 0) = (2, 3, 4)$, $T(0, 1, 0) = (1, 2, 3)$, $T(0, 0, 1) = (1, 1, 1)$. Find the matrix T relative to the ordered basis $(\epsilon_1, \epsilon_2, \epsilon_3)$ where $\epsilon_1 = (1,0,0)$, $\epsilon_2 = (0,1,0)$, $\epsilon_3 = (0,0,1)$. Deduce that T is not invertible.

3. Answer any *two* questions : 10×2=20

(a) (i) State and prove the necessary and sufficient condition for a nonempty subset of a vector space $V(F)$ is subspace of $V(F)$. 4

(ii) Let $W_1 = L\{(1, -2, -1), (2, 3, 5)\}$ and $W_2 = L\{(1, -2, 0), (3, -3, 0)\}$, then show that W_1 and W_2 are two subspaces of $V_3(R)$. Determine $\dim W_1$, $\dim W_2$, $\dim W_1 \cap W_2$ and hence deduce that $\dim(W_1 + W_2) = 3$. 2+4

(b) (i) Let V is a finite dimensional vector space over the field F. Define null space and rank (T) for a linear transformation $T: V(F) \to W(F)$. Prove that rank (T) + *nullity* (T) = dim V. 2+5

(ii) Find a basis for the null space of the matrix

$$\begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix}.$$ 3

(c) (i) Prove that $\ker T$ of a linear mapping $T: V \to W$ is subspace of V. 3

(ii) Find the linear mapping $T: R^3 \to R^3$ such that $Im(T)$ is the subspace $W = \{(x_1, x_2, x_3) \in R^3 : x_1 + x_2 - x_3 = 0\}$ in R^3. 3

(iii) Let V be a vector space of all $n\times n$ matrices over the field F, and let B be a fixed $n\times n$ matrix. If $T(A)=AB-BA$ verify that T is a linear transformation from V into V. 4

(d) (i) Define isomorphism for a linear transformation $T:V\rightarrow W$. Let V and W be finite dimensional vector spaces over the field F. Prove that V and W are isomorphic iff dim $V=$ dim W. 2+4

(ii) Let (x, y, z) be an ordered basis of a real vector space $V(F)$ and a linear mapping $T:V\rightarrow V$ is defined by $T(x)=x+y+z$, $T(y)=x+y$, $T(z)=x$. Find the matrix of T^{-1}. 4

OR

[Vector Calculus and Analytical Geometry]

1. Answer any *ten* from the following questions : 2×10=20

(a) Determine the points of intersection of the line $\frac{x+2}{-1}=\frac{y+4}{-2}=\frac{z-3}{1}$ and the cylinder $x^2+z^2=1$.

(b) Find the length of the chord intercepted by the parabola $y^2=4ax$ on the straight line $y=3x+2$.

(c) Find the eccentric angle of a point on the ellipse $x^2+3y^2=6$ at a distance 2 units from the center.

(d) Prove that the parametric equations $x=\frac{a}{2}\left(\lambda+\frac{1}{\lambda}\right)$ and $x=\frac{b}{2}\left(\lambda-\frac{1}{\lambda}\right)$ represent a hyperbola.

(e) What type of conic does $x^2+2y^2=3$ represent?

(f) Show that $[\alpha+\beta,\beta+\gamma,\gamma+\alpha]=2[\alpha,\beta,\gamma]$, where α,β,γ are any three vectors.

(g) Show that the vectors $A=2i-j+k$, $B=i-3j-5k$, $C=3i-4j-4k$, form the sides of a right-angled triangle.

(h) If $\vec{a}, \vec{b}, \vec{c}$ are three non-zero vectors, no two of them are collinear and the vector $\vec{a}+\vec{b}$ is collinear with $\vec{c}$; $\vec{b}+\vec{c}$ is collinear with $\vec{a}$, then $\vec{a}+\vec{b}+\vec{c}=\vec{0}$.

(i) If PSP' be the focal chord of a conic $\frac{l}{r}=1-e\cos\theta$, then show that $\frac{1}{SP}+\frac{1}{SP'}=\frac{2}{l}$.

(j) If $\vec{a}=(1,0,5)$. Find the unit vector of $\vec{a}$.

(k) If $\vec{r}=x\vec{i}+y\vec{j}+z\vec{k}$ and $|\vec{r}|=r=\sqrt{x^2+y^2+z^2}$, then prove that $\vec{\nabla}(f(r))=f'(r)\vec{\nabla}r$.

(l) Find the equation of the sphere described on the join of $P(2, -3, 4)$ and $Q(-1, 0, 5)$ as diameter.

(m) Prove that $\vec{\nabla}.(\vec{\nabla}\times\vec{f})=0$.

(n) The eccentricities of the hyperbolas $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and $\frac{x^2}{b^2}-\frac{y^2}{a^2}=1$ are e and e' respectively. Prove that $\frac{1}{e^2}-\frac{1}{(e')^2}=1$.

(o) $\vec{a}=2\hat{i}+3\hat{j}-\hat{k}$ and $\vec{b}=6\hat{i}+9\hat{j}-3\hat{k}$. Then find $\vec{a}\times\vec{b}$.

2. Answer any *four* from the following questions : 5×4=20

(a) Show that the locus of the pole w.r.t. the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ of any tangent to the director circle of the ellipse $\frac{x^2}{c^2}+\frac{y^2}{d^2}=1$ is $\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{c^2+d^2}$.

(b) Prove that $\vec{F}=(y^2\cos x+z^3)\vec{i}+(2y\sin x-4)\vec{j}+(3xz^2+2)\vec{k}$ is a conservative force field. Find the scalar potential for $\vec{F}$. Also, find the work done in moving a particle in this field form $(0, 1, -1)$ to $\left(\frac{\pi}{2},-1,2\right)$.

(c) Prove that an angle inscribed in a semi-circle is a right angle.

(d) Show that the vector $\frac{1}{r^3}\vec{r}$ is solenoidal where $\vec{r}=x\vec{i}+y\vec{j}+z\vec{k}$ and $|\vec{r}|=r$.

(e) Find the equation of the sphere for which the circle $x^2+y^2+z^2+7y-2z+2=0$, $2x+3y+4z=8$ is a great circle.

(f) Determine the locus of the points of intersection of perpendicular generators of the paraboloid

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=2z.$$

3. Answer any ***two*** questions : 10×2=20

(a) (i) If $\left|\vec{a}+\vec{b}\right|=\left|\vec{a}-\vec{b}\right|$, then prove that the vectors $\vec{a}$ and $\vec{b}$ are orthogonal. 5

(ii) If $\vec{r}=(a\cos t)\vec{i}+(a\sin t)\vec{j}+(at\tan\alpha)\vec{k}$,

then show that $\left|\frac{d\vec{r}}{dt}\times\frac{d^2\vec{r}}{dt^2}\right|=a^2\sec\alpha$. 5

(b) (i) Establish the following relation :

$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}\right)=-\vec{\nabla}^2\vec{A}+\vec{\nabla}\left(\vec{\nabla}.\vec{A}\right)$. 5

(ii) If θ is the angle between the surfaces $xy^2z=3x+z^2$ and $3x^2-y^2+2z=1$ at the point (1, –2, 1), show that $\cos\theta=\frac{3}{7\sqrt{6}}$. 5

(c) (i) Tangent are drawn to the parabola $y^2=4ax$ at the points whose abscissa are in the ratio $p:1$. Show that the locus of their point of intersection is a parabola. 8

(ii) Find a generator of the cone $5yz8xz+xy=0$. 2

(d) A sphere of constant radius r passes through the origin and cuts the axes in A, B, C. Prove that the locus of the foot of the perpendicular from origin to the plane ABC is given by

$\left(x^2+y^2+z^2\right)^2\left(x^{-2}+y^{-2}+z^{-2}\right)=4r^2$. 10
