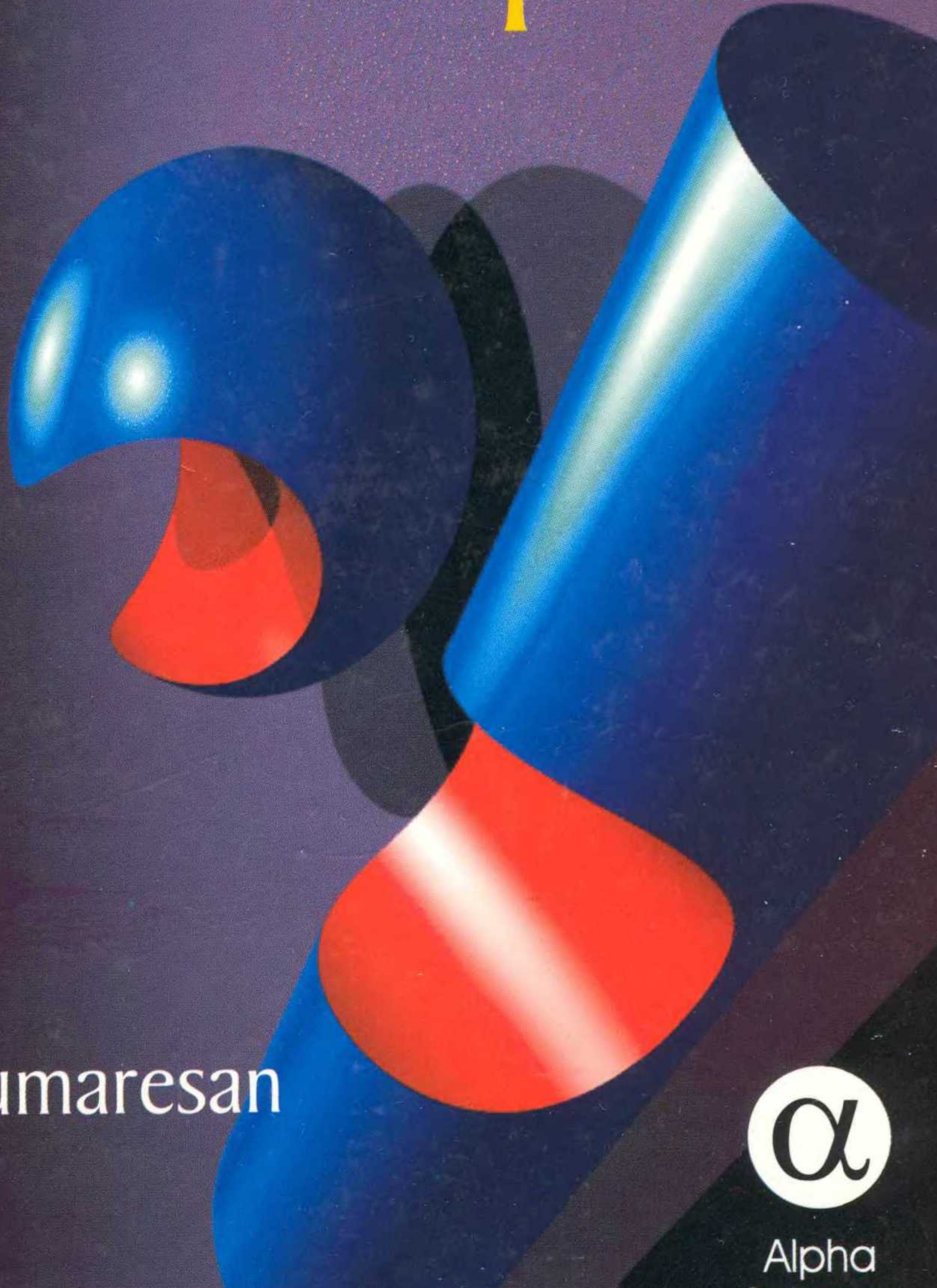
Topology of Metric Spaces



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Alpha Science

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Printed from the camera-ready copy provided by the Author.

ISBN 1-84265-250-8

Printed in India.

Dedicated to All my Teachers

especially to

N. Ganesan
R. Balakrishnan
V. Ganapathy Iyer
M.S. Raghunathan
M.S. Narasimhan
K. Okamoto
R. Parthasarathy

Preface

The aim is to give a very streamlined development of a course in metric space topology emphasizing only the most useful concepts, concrete spaces and geometric ideas. To encourage the geometric thinking, I have chosen large number of examples which allow us to draw pictures and develop our intuition and draw conclusions, generate ideas for proofs. To this end, the book boasts of a lot of pictures. A secondary aim is to treat this as a preparatory ground for a general topology course and arm the reader with a repertory of examples. To achieve this, I have adopted the following strategy. Whenever a definition makes sense in an arbitrary topological space or whenever a result is true in an arbitrary topological space, I use the convention ... (metric) space .... This is to tell the reader the sentence makes mathematical sense in any topological space and if the reader wishes, he may assume that the space is a metric space. See, for example, Def. 4.1.3, Ex. 4.4.12, Def. 5.1.1 and Theorem 5.1.31. On few occasions, I have also shown that if we want to extend the result from metric spaces to topological spaces, what kind of extra conditions need to be imposed on the topological space. These instances may give the students an idea of why various special types of topological spaces are introduced and studied. A third aim is to use this course as a surrogate for real analysis to reinforce some of the concepts from basic analysis while dealing with examples such as functions spaces. This also helps the students to gain some perspective of modern analysis.

The discerning experts will perceive that I have preferred definitions that use only the primitive concept of an open ball (or open sets) rather than secondary concepts. (See, for instance, the definitions of open sets, bounded sets, dense sets etc.) Teachers who use this book should feel free to use their discretion to change the order of the topics. I usually prefer to introduce continuity as early as possible, immediately after open sets and convergent sequences are introduced. In fact, I rarely follow the sequence in which the topics appear here. I am of the opinion that open sets, convergence of sequences, continuity, compactness, connectedness and completeness should be the focus of attention and nucleus of the course. All other concepts may be developed as and when necessary. Another important suggestion is "Draw pictures". After all, topology is geometric in nature and most of the proofs can be better explained by drawing pictures.

I also suggest that the proofs of the theorem should be explained in detail with appropriate pictures with minimal writing on the board. The students should be asked to write the proofs on their own immediately after this.

I have included many 'easy to see' remarks and observations as exercises. Most of them may get a passing mention during a course, but students may not appreciate them at the first instance. If these observations are given as exercises, the students may think over them and retain them for later use. Also, I have included many pictures in the book. I have seen students' classroom notebooks containing no pictures in a year long course in topology! Many a time, I have shown how a picture may lead to an idea of a solution. This, hopefully, will encourage the students to think geometrically. The book is written with a view towards self-study by the students. To make them take active participation in the learning of the subject, very often I spell out the strategy of the proof and encourage the reader to complete it on his own. This also shows the students how even long proofs have a simple core idea which evolves into a more complete and precise proof. This will help them not to miss the wood for the trees. Also, many results are broken into simple exercises and they precede the results. Another noteworthy feature of this book is that almost all results are followed by typical applications.

Some of the topics/results could be assigned for student seminars. A few of them are: Tietze extension theorem, Existence of nowhere differentiable but everywhere continuous functions, Picard's existence theorem, topologist's sine curve, Arzela-Ascoli theorem, Connectedness and path-connectedness of S^n etc.

The book may also be used as a supplementary text for courses in general (or point-set) topology so that students will acquire a lot of concrete examples of spaces and maps.

A prerequisite for the course is an introductory course in real analysis.

The book contains approximately 400 exercises of varying difficulty. I have three governing principles when I assign exercises to the students: (1) There should be a reasonable drill by means of straightforward exercises to test and consolidate the understanding of the definitions and results, (2) standard and typical applications should be given so that in future students will be able to perceive what results may be used to solve a problem and (3) mildly challenging to challenging exercise so that students should not think that everything will fall into their lap. They are also meant to build their perseverance. Most of the exercises are given copious hints. Students are advised to read the hint only after pondering over the problem for sometime.

It is my pleasure to thank the participants of M.T.&.T.S. Programme on whom most of this material was tested. Their enthusiasm and their submission of compilation of the exercises and examples given during my courses were added to this book. I thank A.V. Jayanthan, Goa University who went through a preliminary version of the manuscript

and pointed out the typos and egregious errors and made suggestions for improvement. Special thanks are to Ajit Kumar who made sense out of my rough sketches, applied his own imagination and drew all the figures in this book. But for his contributions, the book will lack one of the noteworthy features of the book, namely, profusion of figures.

I also take this opportunity to thank the charged atmosphere that prevailed in my department during the month of April 2004. It was this which really propelled me into taking up the project of writing this book, since I wanted to work on something that I enjoy and that will help me preserve my sanity. (Yes, it is a case of blatant escapism!)

A Few words to the students. This book is written keeping in mind the difficulties of the beginners in the subject and with a view towards self-study. I have motivated the concepts geometrically as well as using real-life examples/analogies whenever possible. Most of the common mistakes or misconceptions are pointed out. Lots of easy exercises are given so as to consolidate your understanding. You should try to solve at least half of them. It is my fervent hope that the book will encourage you to think geometrically, put some of the basic tricks, results and examples at your disposal for your future endeavour.

A teacher's manual containing more detailed hints and solutions to most of the exercises is under preparation. The interested teacher may contact me on email and receive a pdf version in the near future.

I would like to receive suggestions for improvement, corrections and critical reviews at *kumaresa@sankhya.mu.ac.in*

Mumbai S. Kumaresan

Contents

Chapter 1

Basic Notions

1.1 Definition and Examples

A metric space is a set in which we can talk of the distance between any two of its elements. The definition below imposes certain natural conditions on the distance between the points.

Definition 1.1.1. Let X be a nonempty set. A function $d\colon X \times X \to \mathbb{R}$ is said to be a *metric* or a *distance function* on X if d satisfies the following properties:

(i) $d(x, y) \geq 0$ for all $x, y \in X$ and $d(x, y) = 0$ iff $x = y$.

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$.

(iii) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$. This is known as the *triangle inequality.*

The pair (X, d) is then called a *metric space.*

Example 1.1.2. The most important example is the set $\mathbb{R}$ of real numbers with the metric $d(x, y) := |x - y|$. Recall the *absolute value* of a real number: $|x| = \begin{cases} x & \text{if } x > 0 \\ -x & \text{if } x < 0. \end{cases}$

Observe that

$$x \leq |x| \text{ and } -x \leq |x| \text{ for } x \in \mathbb{R}. \tag{1.1}$$

It is easy to see that d satisfies the first two conditions of the metric. The triangle inequality follows form the triangle inequality of the absolute value:

$$|x + y| \leq |x| + |y| \text{ for all } x, y \in \mathbb{R}. \tag{1.2}$$

Let us quickly review a proof assuming the order relation on $\mathbb{R}$:

Case 1: Let $|x + y| = x + y$. Then $|x + y| = x + y \leq |x| + |y|$ by (1.1).

Case 2: Let $|x+y| = -(x+y)$. We have $|x+y| = -x-y \leq |x|+|y|$ by (1.1).

We have completed the proof of the triangle inequality (1.2) for the absolute value. Also, note that the equality occurs in (1.2) iff x and y are both nonnegative or both non-positive. Assume that the equality occurs in the triangle inequality. Let us further assume that Case 2 occurs. Then $|x+y| = -x-y = |x|+|y|$ holds so that $(|x|+x)+(|y|+y) = 0$. The terms on the left side of this equation are nonnegative so we conclude that $|x| = -x$ and $|y| = -y$. Hence both x and y are nonpositive. Similar analysis of Case 1 yields that both x and y are nonnegative.

It is now an easy matter to derive the triangle inequality for d:

$$
\begin{aligned}
d(x,z) &= |x-z| \\
&= |(x-y)+(y-z)| \\
&\leq |x-y|+|y-z| \quad \text{(by triangle inequality for } |\ |) \\
&= d(x,y)+d(y,z).
\end{aligned}
$$

We refer to d as the absolute value metric.

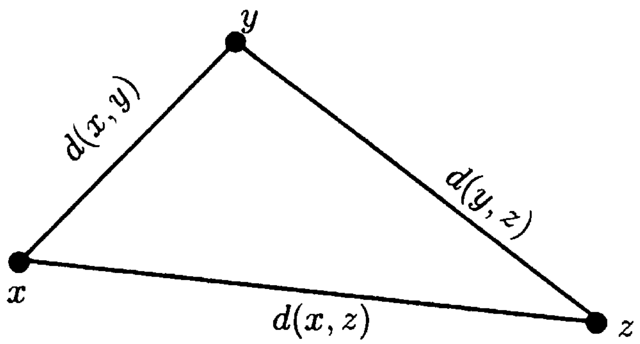

Figure 1.1: Triangle Inequality

Ex. 1.1.3. We may also prove (1.2) as follows: Observe that $x \leq |x|$ for all $x \in \mathbb{R}$ and that $|x| = \sqrt{x^2}$, the nonnegative square root of x^2. Therefore,

$$|x+y|^2 = (x+y)^2 = x^2+y^2+2xy \leq |x|^2+|y|^2+2\,|x|\,|y| = (|x|+|y|)^2.$$

(We used the fact that $|xy| = |x|\,|y|$ for all real numbers x and y in the proof above.) Since $t \to \sqrt{t}$ is increasing in $(0,\infty)$, the result follows.

Ex. 1.1.4. When does equality hold in the triangle inequality for the absolute value metric in $\mathbb{R}$? (See Theorem 1.1.21 for a more general case.)

Example 1.1.5. We now define the absolute value of a complex number and use it to define a metric on $\mathbb{C}$.

For $z \in \mathbb{C}$, we define $|z| = \sqrt{x^2+y^2}$ if $z = x+iy$, $x, y \in \mathbb{R}$. We write $\operatorname{Re} z$ (respectively, $\operatorname{Im} z$) for the real (respectively, the imaginary) part of the complex number.

We observe the following facts about the absolute value function on $\mathbb{C}$:

1. $|z| = |\bar{z}|$ for $z \in \mathbb{C}$.
2. $|z|^2 = z\bar{z}$ for $z \in \mathbb{C}$.
3. $\operatorname{Re} z \leq |z|$ and $\operatorname{Im} z \leq |z|$ for $z \in \mathbb{C}$.
4. $|zw| = |z|\,|w|$ for $z, w \in \mathbb{C}$.
5. For $z, w \in \mathbb{C}$, we have the triangle inequality: $|z+w| \leq |z| + |w|$.

We leave the verification of 1–4 to the reader. We prove 5. We write $\operatorname{Re} z$ for the real part x of the complex number z. We have

$$
\begin{aligned}
|z+w|^2 &= (z+w)\overline{(z+w)} \\
&= |z|^2 + |w|^2 + z\bar{w} + \bar{z}w \\
&= |z|^2 + |w|^2 + 2\operatorname{Re} z\bar{w} \\
&\leq |z|^2 + |w|^2 + 2\,|z\bar{w}| \\
&= |z|^2 + |w|^2 + 2\,|z|\,|w| \\
&= (|z| + |w|)^2.
\end{aligned}
$$

From this, the triangle inequality follows.

Given $z, w \in \mathbb{C}$, we define $d(z, w) := |z - w|$. It is now easy to show that d is a metric on $\mathbb{C}$.

Example 1.1.6 (Discrete Metric). Let X be a nonempty set. Define $d(x, y) = 0$ if $x = y$ and $d(x, y) = 1$ if $x \neq y$. We leave the easy exercise of showing that d is a metric on X to the reader. The metric d called the *discrete metric.*

Example 1.1.7. Let $V = \mathbb{R}^n$. The following are metrics on $\mathbb{R}^n$:

(a) $d_1(x, y) := \sum_{k=1}^n |x_k - y_k|$.

(b) $d_\infty(x, y) := \max\{|x_k - y_k| : 1 \leq k \leq n\}$.

We leave the easy verifications to the reader. We shall come back to them later from a different perspective. See Ex. 1.1.22.

In U.S.A., the metric d_1 is known as the taxi-cab metric. Can you see why? Draw pictures especially the lattice of points with integer coordinates in $\mathbb{R}^2$ and see what the d_1-distance between $(1, 2)$ and $(-4, 8)$ is. See Figure 1.2.

Definition 1.1.8. An *inner product* on a real vector space V is a map

$$\langle\ ,\rangle : V \times V \to \mathbb{R}$$

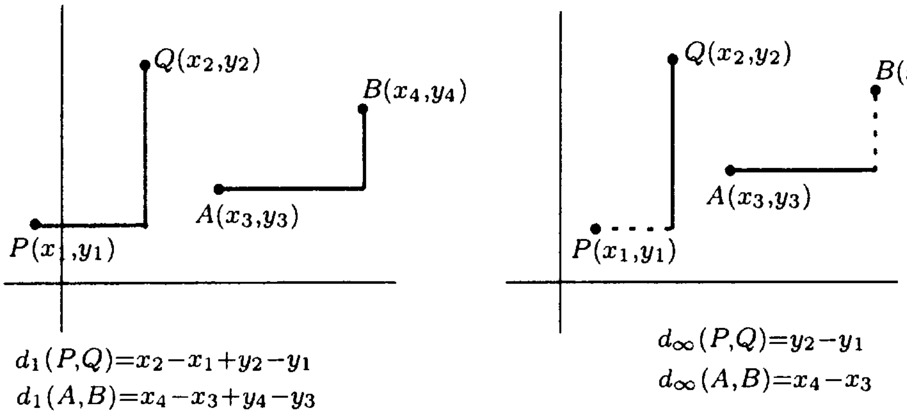

Figure 1.2: d_1 and d_∞ metrics

satisfying the following properties: For $x, y, z \in V$ and $\alpha \in \mathbb{R}$,

(a) $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0$ if and only if $x = 0$,

(b) $\langle x, y\rangle = \langle y, x\rangle$,

(c) $\langle x + z, y\rangle = \langle x, y\rangle + \langle z, y\rangle$ and $\langle x, y + z\rangle = \langle x, y\rangle + \langle x, z\rangle$,

(d) $\langle \alpha x, y\rangle = \alpha \langle x, y\rangle$.

$(V, \langle\ ,\rangle)$ is called an *inner product space*. For brevity sake, we may say V is an inner product space without explicitly mentioning the inner product $\langle\ ,\ \rangle$.

Example 1.1.9. Consider $V = \mathbb{R}^n$. If $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ are in $\mathbb{R}^n$, then their dot product $\langle x, y\rangle$ (or, $x \cdot y$) is defined as $\langle x, y\rangle = \sum_{i=1}^n x_i y_i$. Then it is easy to check that the dot product is an inner product. $(\mathbb{R}^n, \langle,\rangle)$ is called the n-dimensional Euclidean space.

Example 1.1.10. Let $\mathcal{C}[0, 1]$ denote the vector space of all real valued continuous functions on $[0, 1]$. If $f, g \in \mathcal{C}[0, 1]$ define $\langle f, g\rangle = \int_0^1 f(t)g(t)\,dt$, where the integral is in the Riemann sense. Note first of all that the integral exists (thanks to analysis!). The crucial thing to show is that $\langle f, f\rangle = 0$ if and only if $f = 0$. This follows from Lemma 1.1.11 below. The rest of the properties follow from well-known properties of the (Riemann) integral. Thus $(\mathcal{C}[0, 1], \langle,\rangle)$ is an inner product space.

Lemma 1.1.11. *Let $f\colon [0, 1] \to \mathbb{R}$ be continuous with $f(t) \geq 0$ for $t \in [0, 1]$. Then $\int_0^1 f(t)\,dt = 0$ if and only if $f(t) = 0$ for all $t \in [0, 1]$.*

Proof. To prove the nontrivial part, assume that $\int_0^1 f(t)\,dt = 0$. If f is not identically 0, since $f \geq 0$, there exists t_0 such that $f(t_0) > 0$. Let $\alpha := f(t_0)$ and $\varepsilon := \alpha/2$. For this value of ε, by continuity of f at t_0, there is a δ such that $f(t) \in (\frac{\alpha}{2}, \frac{3\alpha}{2})$ for $t \in (t_0 - \delta, t_0 + \delta)$. Using various properties of the integral, we see that

$$\int_0^1 f(t)\,dt \geq \int_{t_0-\delta}^{t_0+\delta} f(t)\,dt \geq \int_{t_0-\delta}^{t_0+\delta} \frac{\alpha}{2}\,dt = \alpha\delta > 0.$$

This contradicts our assumption that $\int_0^1 f(t)\,dt = 0$.

Question: Can you spell out the properties of the (Riemann) integral used while deriving the displayed inequality? □

Remark 1.1.12. Note that if we assume that f is non-negative and Riemann integrable on $[0,1]$ such that $\int_0^1 f(t)\,dt = 0$, then we cannot conclude that $f = 0$ on $[0,1]$. For instance, consider the function $f(t) = 0$ if $t \neq 1/2$ and $f(1/2) = 10$. Then f is Riemann integrable on $[0,1]$ and $\int_0^1 f(t)\,dt = 0$.

Definition 1.1.13. Let V be an inner product space. Given a vector $x \in V$, we define the *norm* or length $\|x\|$ (read as norm of x) as the nonnegative square root of $\langle x, x\rangle$, that is, by $\|x\| := \sqrt{\langle x, x\rangle}$.

The most important examples are the Euclidean spaces. On the vector space $\mathbb{R}^n$, we define the inner product $\langle x, y\rangle := \sum_{j=1}^n x_j y_j$. Note that when $n = 2$, $\|(x,y)\| = \sqrt{x^2+y^2}$ is the length of the vector (x,y).

The norm function $\| \ \| : V \to \mathbb{R}$ has the following properties:

(1) $\|x\| \geq 0$ for all $x \in V$ and $\|x\| = 0$ if and only if $x = 0$.

(2) $\|\alpha x\| = |\alpha|\,\|x\|$, $x \in V$ and $\alpha \in \mathbb{R}$.

Furthermore, given a nonzero vector $v \in V$, the vector defined by $u := \frac{v}{\|v\|}$ is such that $\|u\| = 1$ and $v = \|v\|\,u$. This u is called the *unit vector* along v. (See Figure 1.3.) In general, we say that a vector $x \in V$ is of unit norm if $\|x\| = 1$.

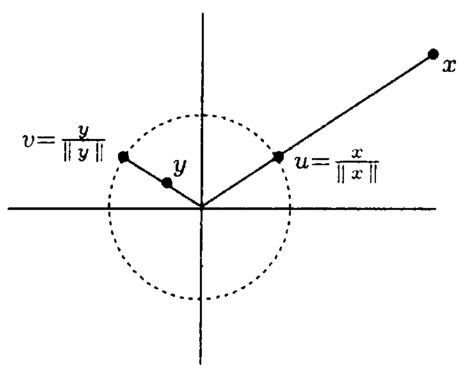

Figure 1.3: Unit vectors along x and y

Theorem 1.1.14 (Cauchy-Schwarz Inequlaity). *Let V be an inner product space. Then we have*

$$|\langle x, y\rangle| \leq \|x\| \, \|y\| \text{ for all } x, y \in V.$$

The equality holds iff one of them is a scalar multiple of the other.

Proof. If $x = 0$ or $y = 0$, then $\langle x, y\rangle = 0$ and either $\langle x, x\rangle = 0$ or $\langle y, y\rangle = 0$. Hence the result. Now consider the case when $\|x\| = \|y\| = 1$. Consider $\langle x - y, x - y\rangle$. Then

$$\begin{aligned} 0 \leq \langle x - y, x - y\rangle &= \langle x, x\rangle + \langle y, y\rangle - 2\langle x, y\rangle \\ &= 2 - 2\langle x, y\rangle \quad \text{as } \|x\| = \|y\| = 1. \end{aligned}$$

Thus we conclude that $2 - 2\langle x, y\rangle \geq 0$ or $\langle x, y\rangle \leq 1$.

Similarly $\langle x + y, x + y\rangle \geq 0$ yields $-\langle x, y\rangle \leq 1$. Hence

$$|\langle x, y\rangle| \leq 1 = \|x\| \, \|y\| . \tag{1.3}$$

We now prove the statement concerning the equality. Let $|\langle x, y\rangle| = 1$. Then either $\langle x, y\rangle = 1$ or -1. If $\langle x, y\rangle = 1$, from the above chain of inequalities we deduce that $\langle x - y, x - y\rangle = 0$ or $x = y$. If $\langle x, y\rangle = -1$, we see that $x = -y$. Thus equality holds if and only if either $x + y = 0$ or $x - y = 0$, that is, if and only if $x = \pm y$.

Now suppose x and y are nonzero (not necessarily of unit length). Then $u = \frac{x}{\|x\|}$ and $v = \frac{y}{\|y\|}$ are of unit length. By the previous case $|\langle u, v\rangle| \leq 1$. Therefore,

$$\left|\left\langle \frac{x}{\|x\|}, \frac{y}{\|y\|}\right\rangle\right| = \left|\frac{1}{\|x\|}\frac{1}{\|y\|}\langle x, y\rangle\right| \leq 1.$$

From this we get $|\langle x, y\rangle| \leq \|x\| \, \|y\|$.

If x and y are nonzero, then the equality means $\langle x, y\rangle = \|x\| \, \|y\|$ or $-\langle x, y\rangle = \|x\| \, \|y\|$. Assume the first happens.
Then

$$\begin{aligned} \langle x, y\rangle = \|x\| \, \|y\| &\iff \left\langle \frac{x}{\|x\|}, \frac{y}{\|y\|}\right\rangle = 1 \\ &\iff \frac{x}{\|x\|} = \frac{y}{\|y\|} \\ &\iff x = \frac{\|x\|}{\|y\|} y. \end{aligned}$$

The other case is similar. $\square$

Theorem 1.1.15. *The norm* $\| \ \|$ *associated to an inner product on a vector space* V *as in the definition above satisfies the following:*

(i) $\|x\| \geq 0$ *for all* $x \in V$ *and* $\|x\| = 0$ *iff* $x = 0$.

(ii) $\|ax\| = |a| \, \|x\|$ *for any scalar* a *and* $x \in V$.

(iii) $\|x + y\| \leq \|x\| + \|y\|$ *for all* $x, y \in V$. (This is known as the triangle inequality of the norm.)

Proof. We leave the proof of the fact that the norm satisfies the first two conditions as easy exercise. To prove the triangle inequality, we proceed as follows:

$$\begin{aligned} \|x + y\|^2 &= \langle x + y, x + y \rangle \\ &= \langle x, x \rangle + \langle y, y \rangle + 2 \langle x, y \rangle \\ &\leq \|x\|^2 + \|y^2\| + 2 \|x\| \, \|y\| \\ &= (\|x\| + \|y\|)^2. \end{aligned}$$

Since $x \mapsto x^2$ is an increasing function on $[0, \infty)$, we deduce the required inequality. □

Definition 1.1.16. Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$. A *norm* on V is a function $\| \ \| : V \to \mathbb{R}$ satisfying the conditions (i)–(iii) listed in the last theorem.

The pair $(V, \| \ \|)$ is called a *normed linear space*, or NLS in short.

Lemma 1.1.17. *Given an NLS* $(V, \| \ \|)$, *we define* $d(x, y) := \|x - y\|$. *Then* d *is a metric on* V.

Proof. We show that d satisfies the triangle inequality. Let us write $x - z = (x - y) + (y - z)$ and apply the triangle inequality of the norm:

$$\begin{aligned} d(x, z) &= \|x - z\| \\ &= \|(x - y) + (y - z)\| \\ &\leq \|x - y\| + \|y - z\| \\ &= d(x, y) + d(y, z). \end{aligned}$$

Thus d defines a metric on V. □

Remark 1.1.18. The metric d defined by $d(x, y) := \|x - y\|$ on an NLS will be referred to as the metric associated with the norm $\| \ \|$. All metric concepts in the sequel concerning an NLS will be with reference to this metric.

Ex. 1.1.19. Metrics induced by norms are translation invariant:

$$d(x + z, y + z) = d(x, y) \text{ for } x, y, z \text{ in an NLS.}$$

The next theorem explains the geometric meaning of the case when the equality occurs in the triangle inequality in the standard Euclidean metric on $\mathbb{R}^n$. This is typical of the equality cases of many inequalities. They are always very special and, more often than not, have a geometric interpretation. We need the following definition.

Definition 1.1.20. Let V be any real vector space and $x, y, z \in V$. We say that the point z lies between the points x and y iff there exists $t \in \mathbb{R}$, $0 \le t \le 1$ such that $z = tx + (1-t)y$.

Theorem 1.1.21. *Let x and y be two points in an inner product space $(V, \langle , \rangle)$ over $\mathbb{R}$. Let $z \in V$. Then the equality holds in the triangle inequality $d(x, y) \le d(x, z) + d(z, y)$ iff the point z lies between the points x and y.*

Proof. Let z lie between the points x and y, say, $z = tx + (1-t)y$ for some $t \in [0, 1]$. Then

$$\begin{aligned} d(x,z) + d(z,y) &= \|x - z\| + \|z - y\| \\ &= \|x - tx - (1-t)y\| + \|tx + (1-t)y - y\| \\ &= \|(1-t)(x-y)\| + \|t(x-y)\|. \end{aligned}$$

Since $t \ge 0$ and $1 - t \ge 0$, it follows that

$$\begin{aligned} d(x,z) + d(z,y) &= (1-t)\|x - y\| + t\|x - y\| \\ &= \|x - y\| \\ &= d(x,y). \end{aligned}$$

Conversely let us assume that $d(x, y) = d(x, z) + d(z, y)$. We know that $\|x - y\| \le \|x - z\| + \|z - y\|$. But we have equality by our assumption. Therefore, it follows from the equality case of triangle inequality (for a norm induced by an inner product) that $x - z = t(z - y)$ for some $t \ge 0$. This shows that $(1 + t)z = x + ty$ and $z = \frac{1}{1+t}x + \frac{t}{1+t}y$. Since, $t \ge 0$, we see that $0 \le \frac{1}{1+t} \le 1$. Therefore, $z = sx + (1 - s)y$ where $s = \frac{1}{1+t}$. That is, z lies between x and y. This completes the proof. ☐

Ex. 1.1.22. Show that the following are norms on $\mathbb{R}^n$:

(a) $\|x\|_1 := \sum_{k=1}^n |x_k|$.

(b) $\|x\|_\infty := \max\{|x_k| : 1 \le k \le n\}$. *Hint:* To prove triangle inequality, observe that $|x_j| \le \|x\|_\infty$ for $x = (x_1, \ldots, x_j, \ldots, x_n) \in \mathbb{R}^n$.

(c) $\|x\|_2 := (\sum_{k=1}^n |x_k|^2)^{1/2}$. This is called the Euclidean norm. It is the norm associated with the dot-product on $\mathbb{R}^n$. In the sequel unless otherwise specified, we shall assume that $\mathbb{R}^n$ is equipped with this norm and the metric induced by this norm will be denoted by d.

The metrics induced by the norms $\| \ \|_1$ and $\| \ \|_\infty$ are respectively the metrics d_1 and d_∞ of Ex. 1.1.7.

We generalize these norms to suitable spaces of functions in the next few examples.

Example 1.1.23. Let X be a nonempty set. Let $B(X)$ be the set of all bounded real (or complex) valued functions. Then

$$\| f \|_\infty := \sup\{|f(x)| : x \in X\}$$

defines a norm on $B(X)$. We shall show that the triangle inequality holds. Let $f, g \in B(X)$ and $x \in X$.

$$\begin{aligned} |f(x) + g(x)| &\leq |f(x)| + |g(x)| \\ &\leq \sup\{|f(t)| : t \in X\} + \sup\{|g(t)| : t \in X\} \\ &= \| f \|_\infty + \| g \|_\infty . \end{aligned}$$

Thus the set of real numbers $\{|f(x) + g(x)| : x \in X\}$ is bounded above by the real number $\| f \|_\infty + \| g \|_\infty$. Hence the supremum of the set, namely, $\| f + g \|_\infty$, must be less than or equal to the upper bound $\| f \|_\infty + \| g \|_\infty$. We let d_∞ denote the metric induced by this norm.

This is similar to the norm $\| \ \|_\infty$ on $\mathbb{R}^n$.

Example 1.1.24. Let $X := [0, 1]$, the closed unit interval. Then

$$\| f \|_1 := \int_0^1 |f(t)| \, dt$$

defines a norm on the set of all continuous real/complex valued functions on $[0, 1]$. This is similar to the norm $\| \ \|_1$ on $\mathbb{R}^n$. We let d_1 denote the distance induced buy this norm.

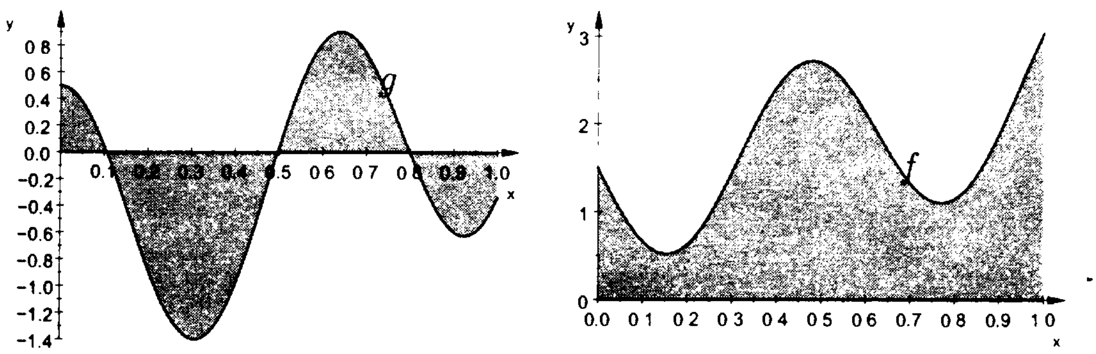

Figure 1.4: Geometric meaning of $\|.\|_1$: area of the shaded region

Here the main problem is to show that if $\|f\|_1 = 0$, then $f = 0$ on [0,1]. But this is already dealt with. See Lemma 1.1.11. The triangle inequality is easy using the properties of the Riemann integral.

$$\begin{aligned}
\|f+g\|_1 &= \int_0^1 |f(t)+g(t)|\,dt \\
&\leq \int_0^1 |f(t)|+|g(t)|\,dt \\
&= \int_0^1 |f(t)|\,dt + \int_0^1 |g(t)|\,dt \\
&= \|f\|_1 + \|g\|_1 .
\end{aligned}$$

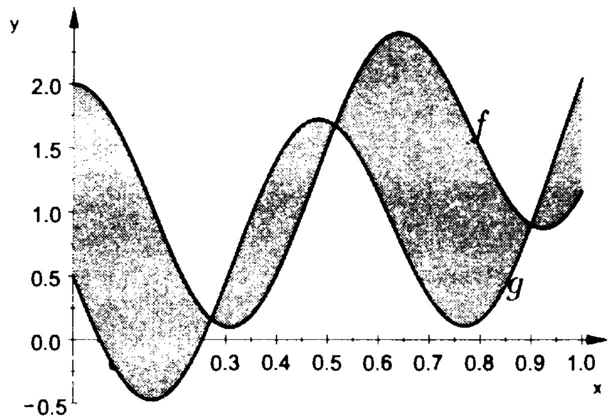

Figure 1.5: Geometric meaning of the d_1-metric: $d_1(f,g)$ is the area of the shaded region.

The geometric meaning of $\|f\|_1$ becomes clear to us if we recall the geometric meaning of the the integral $\int_a^b f(t)\,dt$ of a non-negative function f on an interval $[a,b]$. It is the area of the region bounded by $x=a$, $x=b$, $y=0$ and $y=f(x)$. See Figures 1.4–1.5 where $a=0$ and $b=1$.

Ex. 1.1.25. Let $X := [0,1]$, the closed unit interval. Then

$$\|f\|_2 := \left(\int_0^1 (|f(t)|)^2\,dt\right)^{1/2}$$

defines a norm on the set of all continuous real/complex valued functions on $[0,1]$.

Ex. 1.1.26. Consider the functions $f(t) := t$ and $g(t) = t^2$ for $t \in [0,1]$. Compute $d_1(f,g)$ and $d_\infty(f,g)$.

Ex. 1.1.27. Let $V := C[0,1]$ denote the vector space of all real valued continuous functions on $[0,1]$. Show that $f \mapsto \|f\|_\infty := \sup\{|f(x)| : x \in [0,1]\}$ is a norm on V. (Why does $\|f\|_\infty$ make sense?)

Ex. 1.1.28. Let X be the set of all real sequences. We wish to regard two points (x_n) and (y_n) to be close to each other if their first N terms are equal for some large N. Larger the integer N closer they are. This is achieved by the following definition of the metric:

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ \frac{1}{\min\{i : x_i \neq y_i\}} & \text{if } x \neq y. \end{cases}$$

The triangle inequality $d(x,z) \leq d(x,y) + d(y,z)$ certainly holds if any two of x, y, z are equal. So assume that $x \neq y$, $y \neq z$ and $z \neq x$. Let

$$r := \min\{i : x_i \neq y_i\}, s := \min\{i : y_i \neq z_i\}, t := \min\{i : z_i \neq x_i\}.$$

Clearly $t \geq \min\{r, s\}$ and hence $d(x,z) \leq \max\{d(x,y), d(y,z)\}$.

Ex. 1.1.29. Let (X,d) be a metric space. Let $A \subset X$ be nonempty. Define for $x, y \in A$, $\delta(x,y) := d(x,y)$. Then δ is a metric on A, called the *induced metric* on the subset A.

Ex. 1.1.30. Let d be a metric on X. Define $\delta(x,y) := \min\{1, d(x,y)\}$ for all $x, y \in X$. Show that δ is a metric on X.

Ex. 1.1.31. Let (X,d) be a metric space. Define

$$\delta(x,y) := \frac{d(x,y)}{1 + d(x,y)}, \text{ for all } x, y \in X.$$

Show that δ is a metric on X.

Ex. 1.1.32 (Product Metric). Let (X,d) and (Y,d) be metric spaces. Show that

$$d((x_1,y_1),(x_2,y_2)) := \max\{d(x_1,x_2), d(y_1,y_2)\}$$

defines a metric on the product set $X \times Y$. We refer to the metric on $X \times Y$ as the *product metric*.

Can you think of other metrics on $X \times Y$ coming from the original metrics on X and Y?

Ex. 1.1.33. Suppose (X,d) and (Y,δ) be metric spaces. Is there a metric on $X \cup Y$ which induces d on X and δ on Y? (Assume $X \cap Y = \emptyset$).

Ex. 1.1.34. Let $M(n, \mathbb{R})$ denote the set of all $n \times n$ real matrices. We identify any $A = (a_{ij}) \in M(n, \mathbb{R})$ with the vector

$$(a_{11}, a_{12}, \ldots, a_{1n}, \ldots, a_{n1}, \ldots, a_{nn}) \in \mathbb{R}^{n^2}.$$

This map is a linear isomorphism between $M(n, \mathbb{R})$ and $\mathbb{R}^{n^2}$. Using this linear isomorphism, we define

$$\|A\| := (\sum_{i,j} |a_{ij}|^2)^{1/2} = \|(a_{11}, \ldots, a_{nn})\|.$$

Thus $M(n, \mathbb{R})$ is an NLS.

Lemma 1.1.35 (Young's inequality). *Let x, y be nonnegative real numbers. Let $p > 1$ and q be defined in such a way that $\frac{1}{p} + \frac{1}{q} = 1$ holds. We have the following inequality known as the Young's Inequality:*

$$xy \leq \frac{x^p}{p} + \frac{y^q}{q}. \tag{1.4}$$

Equality holds iff $x^p = y^q$.

Proof. The strategy is as follows. Fix $y > 0$. Consider the function of

$$f(x) := \frac{x^p}{p} + \frac{y^q}{q} - xy \qquad \text{for } x > 0.$$

We apply maxima-minima tests of one variable calculus to arrive at the inequality. The reader should go ahead and complete the proof.

The derivative of f is $f'(x) = x^{p-1} - y$. Therefore, the critical point, that is, the point at which the derivative vanishes, is given by $x_0 = y^{\frac{1}{p-1}}$. Clearly, $f''(x_0) > 0$. We therefore conclude that $f(x_0) = 0$ is the minimum of f on $(0, \infty)$ whence it follows that$f(x) \geq 0 = f(x_0)$. This is the inequality we were after.

Note also that our analysis shows that the equality occurs iff $x = y^{\frac{1}{p-1}}$. Raising to p-th power leads to the result. $\square$

Lemma 1.1.36 (Hölder's inequality). *We let $\mathbb{K}$ stand for $\mathbb{R}$ or $\mathbb{C}$. Let X be $\mathbb{K}^n$ and, for $1 \leq p < \infty$, let $\|x\|_p := (\sum_{i=1}^n |x_i|^p)^{1/p}$ and for $p = \infty$, let $\|x\|_\infty := \max\{|x_i| : 1 \leq i \leq n\}$. For $p > 1$, let q be such that $(1/p) + (1/q) = 1$. For $p = 1$ take $q = \infty$. We have Hölder's inequality:*

$$\sum_i |a_i| |b_i| \leq \|a\|_p \|b\|_q, \text{ for all } a, b \in \mathbb{K}^n. \tag{1.5}$$

Equality holds iff $C_1 |x_k|^p = C_2 |y_k|^q$ for $1 \leq k \leq n$ for some nonzero constants C_1 and C_2.

Proof. The strategy is as follows. We take $x = \frac{|a_i|}{\|a\|_p}$ and $y = \frac{|b_i|}{\|b\|_q}$ in (1.4) and sum over i. Now, it is a straight forward exercise for the reader.

If we take $x = \frac{|a_i|}{\|a\|_p}$ and $y = \frac{|b_i|}{\|b\|_q}$ in (1.4), we get

$$\frac{1}{p}\frac{|a_i|^p}{\|a\|_p^p} + \frac{1}{q}\frac{|b_i|^q}{\|b_i\|_q^q} \geq \frac{|a_i|}{\|a\|_p}\frac{|b_i|}{\|b_i\|_q}. \tag{1.6}$$

Summing this over $i = 1$ to $i = n$, we get

$$\frac{1}{p}\sum_{i=1}^{n}\frac{|a_i|^p}{\|a\|_p^p} + \frac{1}{q}\sum_{i=1}^{n}\frac{|b_i|^q}{\|b\|_q^q} \geq \sum_{i=1}^{n}\left(\frac{|a_i|}{\|a\|_p}\frac{|b_i|}{\|b\|_q}\right).$$

Simplifying we get

$$\frac{1}{p} + \frac{1}{q} \geq \frac{1}{\|a\|_p}\frac{1}{\|b\|_q}\sum_{i=1}^{n}|a_i|\,|b_i|,$$

whence the inequality.

When does equality occur? Going through the proof and recalling when the equality occurs in Young's inequality, we deduce that the equality occurs iff

$$\frac{|x_k|^p}{\|x\|_p} = \frac{|y_k|^q}{\|y\|_q} \iff C_1\,|x_k|^p = C_2\,|y_k|^q \text{ for } 1 \leq k \leq n,$$

for some nonzero constants C_1, C_2. □

Lemma 1.1.37 (Minkowski inequality). *Let $1 \leq p \leq \infty$. We have Minkowski's inequality:*

$$\|a+b\|_p \leq \|a\|_p + \|b\|_p \text{ for } a, b \in \mathbb{K}^n. \tag{1.7}$$

Equality occurs iff there exist constants C_1, C_2 such that $C_1 a = C_2 b$.

Proof. The proof is seen already if $p = \infty$. Therefore we assume that $1 \leq p < \infty$. The case when either $a = 0$ or $b = 0$ is obvious and hence we assume that neither of them is zero.

We again start with a hint and ask the reader to complete the proof on his own. For $1 < p < \infty$, observe

$$\begin{aligned} \sum_i |a_i + b_i|^p &= \sum_i |a_i + b_i||a_i + b_i|^{p-1} \\ &\leq \sum_i |a_i||a_i + b_i|^{p-1} + \sum_i |b_i||a_i + b_i|^{p-1}. \end{aligned} \tag{1.8}$$

We apply Holder's inequality to each of the summands.

Now we carry out the complete argument. Let us observe certain relations between p and q. Since $1/p + 1/q = 1$, we have $p = q(p-1)$ and $1/q = (p-1)/p$. We use them below.

Consider a typical term $\sum_{i=1}^{n} |a_i| \, |a_i + b_i|^{p-1}$. If we apply (1.5) to this sum , we get

$$
\begin{aligned}
\sum_{i=1}^{n} |a_i| \, |a_i + b_i|^{p-1} &\leq \| a \|_p \left[\sum_{i=1}^{n} \left(|a_i + b_i|^{p-1} \right)^q \right]^{1/q} \\
&= \| a \|_p \left[\sum_{i=1}^{n} |a_i + b_i|^{q(p-1)} \right]^{1/q} \\
&= \| a \|_p \sum_{i=1}^{n} |a_i + b_i|^{p/q} \\
&= \| a \|_p \, \| (a+b) \|_p^{p/q} . \qquad (1.9)
\end{aligned}
$$

Similarly for the other term, we have

$$
\sum_{i=1}^{n} |b_i| \, |a_i + b_i|^{p-1} \leq \| b \|_p \, \| (a+b) \|_p^{p/q} . \qquad (1.10)
$$

From (1.8)–(1.10), we get

$$
\| a + b \|_p^p \leq \| a \|_p \, \| a + b \|_p^{p/q} + \| b \|_p \, \| a + b \|_p^{p/q} .
$$

Dividing both sides of the inequality by the positive number $\| a + b \|_p^{p/q}$ and using the fact $p - (p/q) = p(1 - 1/q) = 1$ yields the Minkowski inequality.

Equality case if left to the reader. □

Example 1.1.38 (Sequence Spaces). Let $1 \leq p < \infty$. Let ℓ_p be defined as follows:

$$
\ell_p := \{ (a_n)_{n=1}^{\infty} : a_n \in \mathbb{R} \text{ or } \mathbb{C}, \text{ and } \sum_{n=1}^{\infty} |a_n|^p < \infty \}.
$$

Let ℓ_∞ stand for $(B(\mathbb{N}), \| \ \|_\infty)$. Since $p = \infty$ has already been dealt with in Example 1.1.23, we shall concentrate on the case when $1 \leq p < \infty$.

We first of all show that ℓ_p is a vector space over $\mathbb{R}$ (or $\mathbb{C}$, as the case may be). Let $(x_n) \in \ell_p$ and a be a scalar. Then $ax = (ax_n)$. Clearly, $\sum_{n=1}^{\infty} |ax_n|^p = |a|^p \sum_{n=1}^{\infty} |x_n|^p < \infty$. Therefore, $ax \in \ell_p$. Let $x = (x_n)$ and $y = (y_n) \in \ell_p$. Then we need to show that $\sum_{k=1}^{\infty} |x_k + y_k|^p < \infty$. This is an interesting argument and runs as follows.

Let $\|x\|_p := (\sum_{k=1}^{\infty} |x_k|^p)^{1/p}$. Then using Minkowski inequality, (1.7), we deduce, for each $n \in \mathbb{N}$,

$$\begin{aligned}\left(\sum_{k=1}^{n} |x_k + y_k|^p\right)^{1/p} &\leq \left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p} + \left(\sum_{k=1}^{n} |y_k|^p\right)^{1/p} \\ &\leq \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p} + \left(\sum_{k=1}^{\infty} |y_k|^p\right)^{1/p}\end{aligned}$$

Since the above inequality is true for all n, it follows that[1]

$$\left(\sum_{k=1}^{\infty} |x_k + y_k|^p\right)^{1/p} \leq \|x\|_p + \|y\|_p .$$

Consequently, raising to the power p, we get

$$\sum_{k=1}^{\infty} |x_k + y_k|^p \leq \left(\|x\|_p + \|y\|_p\right)^p$$

We have thus shown ℓ_p is a vector space as well as established that $(a_n) \mapsto \|(a_n)\|_p := (\sum_{n=1}^{\infty} |a_n|^p)^{1/p}$ is a norm on ℓ_p. Hence $(\ell_p, \| \ \|_p)$ is a normed linear space.

Ex. 1.1.39 (An important inequality). In any metric space (X, d), show that $|d(x, z) - d(y, z)| \leq d(x, y)$.

In an NLS $(X, \| \ \|)$, we have $|\|x\| - \|y\|| \leq \|x - y\|$ for any two vectors $x, y \in X$.

1.2 Open Balls and Open Sets

Definition 1.2.1. Let (X, d) be a metric space. Let $x \in X$ and $r > 0$. The subsets

$$B_d(x, r) := \{y \in X : d(x, y) < r\} \text{ and } B_d[x, r] := \{y \in X : d(x, y) \leq r\}$$

are respectively called the *open* and *closed* balls centred at x with radius r with respect to the metric d. We use this notation only when we want to emphasize that the metric under consideration is d. Otherwise, we denote $B_d(x, r)$ by $B(x, r)$ when there is no source for confusion. Similarly $B[x, r]$ will denote $B_d[x, r]$.

[1] Recall that if $\sum_{k=1}^{\infty} \alpha_k$ is a series of positive terms, then it is convergent iff the sequence of partial sums is bounded, in which case, the sum is $\sup_n \sum_{k=1}^{n} \alpha_k$.

Example 1.2.2. Let $\mathbb{R}$ be with the standard metric. Then we claim that $B(x,r) = (x-r, x+r)$. For, if $y \in B(x,r)$ iff $d(x,y) < r$ iff $|x-y| < r$ iff $y \in (x-r, x+r)$.

Example 1.2.3. Let $X = \mathbb{R}^2$ with the Euclidean metric $d = d_2$. Then $B(0,r) = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < r^2\}$. For, $p := (x,y) \in B(0,r)$ iff $d(p,0) < r$ iff $d(p,0)^2 < r^2$ iff $x^2 + y^2 < r^2$. (Why? For non-negative numbers a, b, we have $a < b$ iff $a^2 < b^2$. Since $0 < b^2 - a^2 = (b-a)(b+a)$ and since $b + a > 0$, it follows that $b^2 - a^2$ is positive iff $b - a$ is.)

More generally, if $q = (a,b)$, then $B(q,r)$ is the set of points inside the circle of radius r with centre at (a,b):

$$B(q,r) = \{(x,y) \in \mathbb{R}^2 : (x-a)^2 + (y-b)^2 < r^2\}.$$

Ex. 1.2.4. Show that any open interval (a,b) in $\mathbb{R}$ is an open ball. Is $\mathbb{R}$ an open ball in $\mathbb{R}$?

Ex. 1.2.5. Let (X,d) be a discrete metric space and $x \in X$. Find the following: (a) $B(x,1/2)$, (b) $B(x,3/4)$, (c) $B(x,1)$, (d) $B(x,r), 0 < r \le 1$ and (e) $B(x,r)$, $r > 1$.

Ex. 1.2.6. Let (X,d) be a metric space, $x \in X$ and $0 < r < s$. Show that $B(x,r) \subseteq B(0,s)$. Show that they may be equal even if $r < s$.

Ex. 1.2.7. With the notation of the last exercise, show that $B(x,r) \subseteq B[x,r]$. Can the open balls be equal?

Ex. 1.2.8. If in a metric space we have $B(x,r) = B(y,s)$, does it mean that $x = y$ and $r = \rho$?

Ex. 1.2.9. Draw figures of the open unit ball $B(0,1)$ in the following metric spaces.

(a) $(\mathbb{R}^2, \| \ \|_2)$, (the standard Euclidean norm).
(b) $(\mathbb{R}^2, \| \ \|_1)$, (the L^1-norm).
(c) $(\mathbb{R}^2, \| \ \|_\infty)$, (the max or sup norm).

Check your pictures with those in Figure 1.6.

Definition 1.2.10. Let V be a real vector space, $x, y \in V$. We let $[x,y] := \{(1-t)x + ty : 0 \le t \le 1\}$. The set $[x,y]$ will be called the line segment joining x and y. A subset A of V is said to be *convex* if for any pair $x, y \in A$, the line segment $[x,y] \subset A$.

Ex. 1.2.11. Let $(X, \| \ \|)$ be an NLS. Show that any ball $B(x,r)$ is convex.

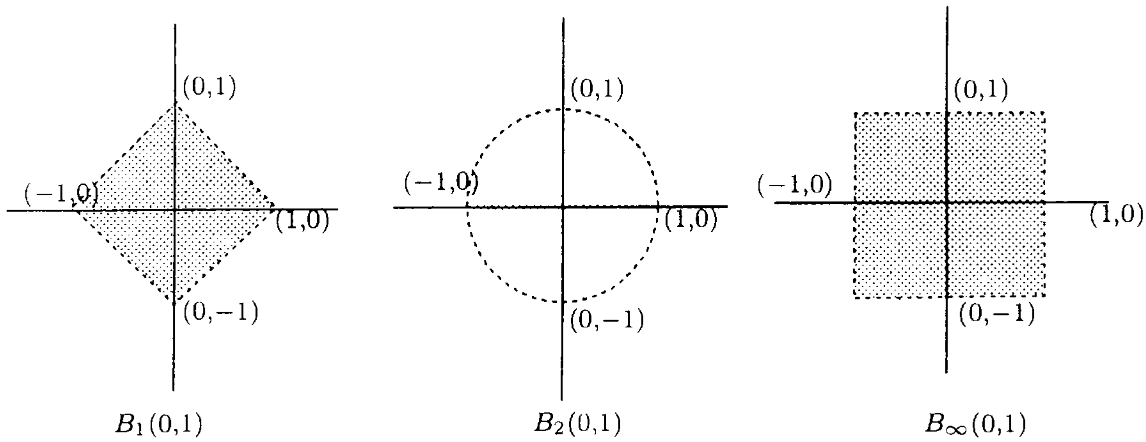

Figure 1.6: Unit Balls with respect to d_1, d_2 and d_∞

Ex. 1.2.12. Show that in an NLS $(X, \| \ \|)$, we have

$$B(x,r) = x + rB(0,1), \quad x \in X, r > 0.$$

Thus if we know the open unit ball, that is, $B(0,1)$ in an NLS, we know all the open balls! To get $B(x,r)$, we dilate (or contract) $B(0,1)$ by r and then translate it by the vector x. See Figure 1.7 on page 17.

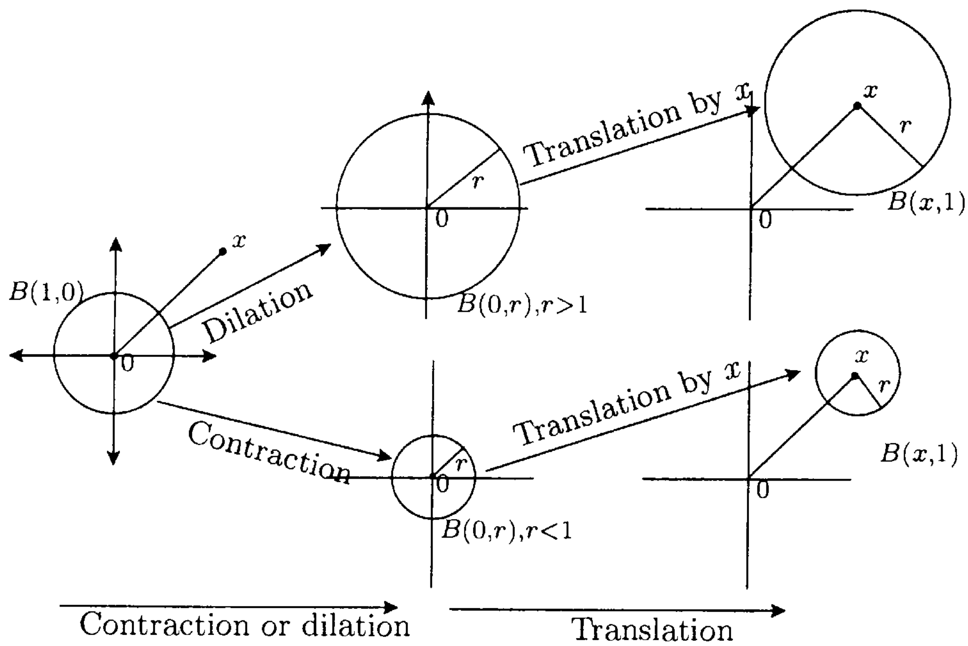

Figure 1.7: Translation and dilation of the unit ball

Ex. 1.2.13. Let $(X, \| \ \|)$ be an NLS, $x \in X$ and $0 < r < s$. Show that $B(x,r) \subsetneqq B(x,s)$. (Compare this with Ex. 1.2.6.) *Hint:* First consider the case $X = \mathbb{R}^2$. Later, in the general case, restrict to the case when $x = 0$.

Ex. 1.2.14. Let $(X, \| \ \|)$ be an NLS, $x \in X$ and $0 < r$. Show that $B(x,r) \subsetneqq B[x,s]$. (Compare this with Ex. 1.2.7.)

Example 1.2.15. Let $X = I = [0,1]$. Let $V := C[0,1]$ be the NLS of continuous real valued functions on $[0,1]$ under the sup norm. How will you visualize $B(0,\varepsilon)$? Let $f \in C[0,1]$. How will you visualize $B(f,\varepsilon)$? See Figure 1.8. An element $\varphi \in C[0,1]$ lies in $B(0,\varepsilon)$ iff its graph lies in the region bounded by the lines $y = \pm\varepsilon$ and $x = 0, 1$. Similarly, φ lies in $B(f,\varepsilon)$ iff its graph lies in the region bounded by the curves $y = f \pm \varepsilon$ and $x = 0, 1$.

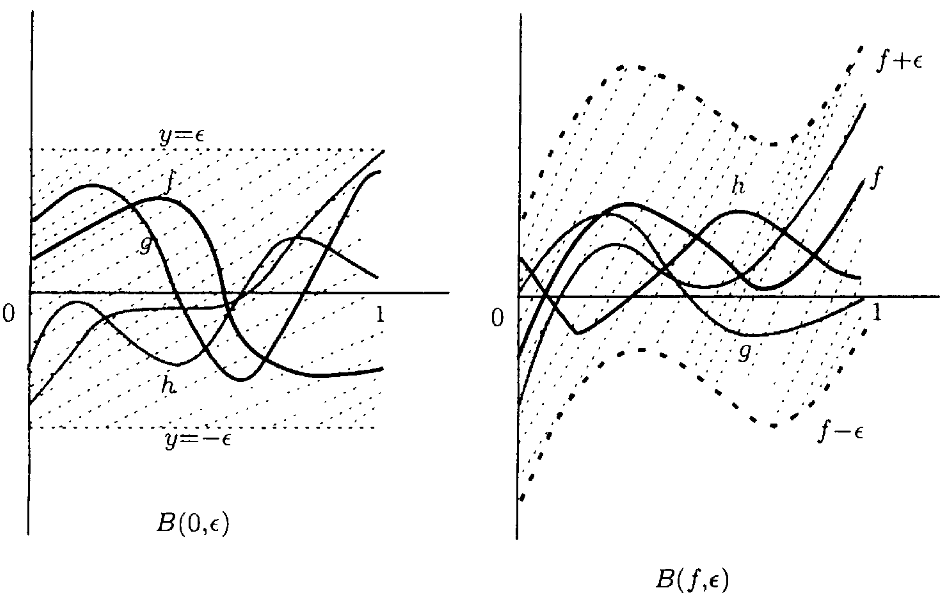

Figure 1.8: Open balls in $C[0,1]$

Ex. 1.2.16. Let $f, g \colon [0,1] \to \mathbb{R}$ be continuous and $f(t) < g(t)$ for all $t \in [0,1]$. Consider the set

$$U := \{h \in C[0,1] : f(t) < h(t) < g(t), \text{ for } t \in [0,1]\}$$

in the space $X := (C[0,1], \| \ \|_\infty)$. Is U a ball in X? If not, can you think of a condition of f and g that will ensure that the set U is an open ball?

Example 1.2.17. Consider the NLS $X = (C[0,1], \| \ \|_1)$, where $\|f\|_1 := \int_0^1 |f(t)|\, dt$. How do we visualize $B(0,1)$ in X?

Recall the geometric meaning of $\|f\|_1$ from Example 1.1.24. Thus, $f \in B(0,1)$ iff the area 'under the graph' of $|f|$ is less than 1. For

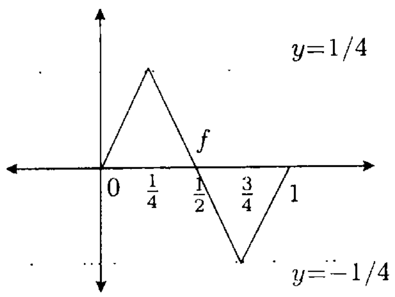

Figure 1.9: f is in $B(0,1)$

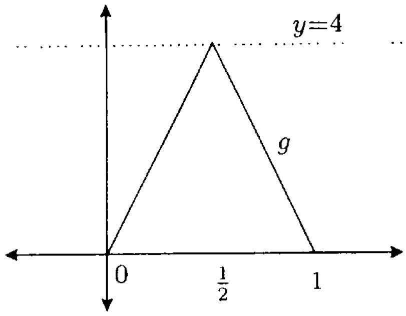

Figure 1.10: g is not in $B(0,1)$

example, f in Figure 1.9 lies in B(0,1) whereas g in Figure 1.10 on page 19 does not.

Ex. 1.2.18 (Hausdorff Property). Given two distinct points $x, y \in X$, there exists $r > 0$ such that $B(x,r) \cap B(y,r) = \emptyset$. (See Figure 1.11.)

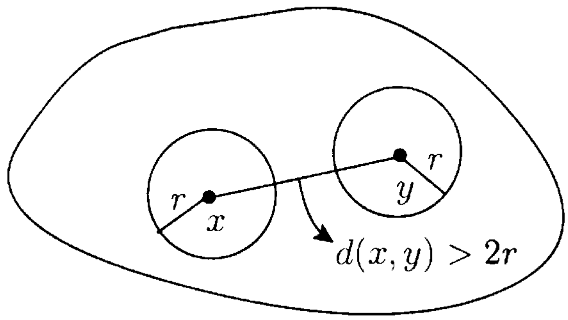

Figure 1.11: Hausdorff Property

Ex. 1.2.19. Let A be a nonempty subset of a metric space (X,d). Let us continue to denote by the same letter d the induced metric on A. Let $B_A(a,r)$ denote the open ball *in the metric space* (A,d) centred at $a \in A$ and radius $r > 0$. Show that $B_A(x,r) = B(x,r) \cap A$, where $B(x,r)$ stands for the open ball in X centred at x and radius r.

Ex. 1.2.20. Let $\mathbb{Z} \subset \mathbb{R}$ be endowed with the induced metric from $\mathbb{R}$. Give a "concrete description" of all open balls in $\mathbb{Z}$. (We are not interested in the description given in the last exercise!)

Definition 1.2.21. A subset $U \subset X$ of a metric space is said to be *d-open* if for each $x \in U$, there exists $r > 0$ such that $B(x,r) \subset U$. See Figure 1.12. If there is no source of confusion about which metric is being used, we shall simply refer to as an open set rather than as d-open set.

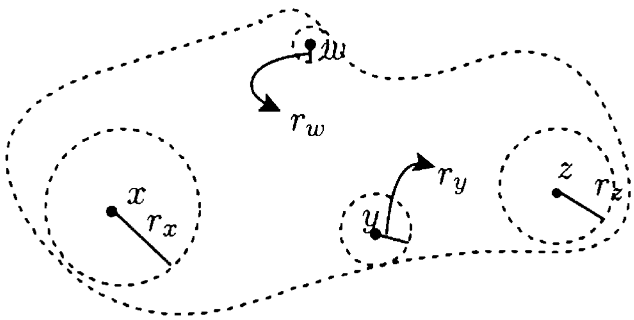

Figure 1.12: Open set in a metric space

Ex. 1.2.22. Find all open sets in a discrete metric space.

Ex. 1.2.23. Prove that an open interval in $\mathbb{R}$ is open.

Lemma 1.2.24. *Let (X, d) be a metric space, $x \in X$ and $r > 0$. Then the open ball $B(x, r)$ is open.*

Proof. Let $y \in B(x, r)$. We need to find $s > 0$ such that $B(y, s) \subset B(x, r)$. Look at Figure 1.13. It suggests us what s could be.

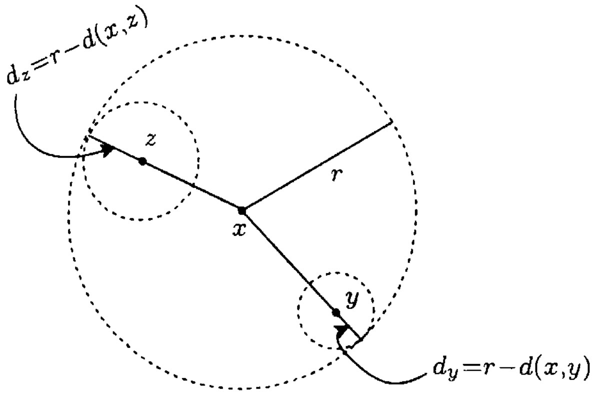

Figure 1.13: $B(x, r)$ is open

Also, let us work backwards. Assume such an s exists. If $z \in B(y, s)$, we want to show that $d(z, x) < r$. Now,

$$d(z, x) \leq d(z, y) + d(y, x) < s + d(x, y) < r.$$

This prompts us to consider $0 < s \leq r - d(x, y)$. Let s be one such. If $z \in B(y, s)$, then we have

$$d(z, x) \leq d(z, y) + d(y, x) < s + d(y, x) < r.$$

Thus, $B(y, s) \subset B(x, r)$. Since y is an arbitrary element of $B(x, r)$, we have proved that $B(x, r)$ is an open set. $\square$

Ex. 1.2.25. Is the empty set $\emptyset \subset X$ open in the metric space (X, d)?

Ex. 1.2.26. Let X be a metric space and $a \in X$. Fix $r > 0$. Show that the set $\{x \in X : d(x, a) > r\}$ is open.

In Exercises 1.2.27–1.2.31, you are required to find "the best possible" r_p such that $B(p, r_p) \subset U$, for $p \in U$, in case you claim that U is open.

Ex. 1.2.27. Let $U := \mathbb{R} \setminus \mathbb{Z}$. Is U open in $\mathbb{R}$?

Ex. 1.2.28. Show that $U := \{(x, y) \in \mathbb{R}^2 : xy \neq 0\}$ is open in $\mathbb{R}^2$. (See Figure 1.14.)

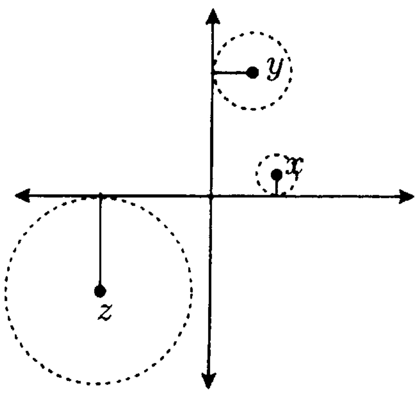

Figure 1.14: $\{xy \neq 0\}$ is open

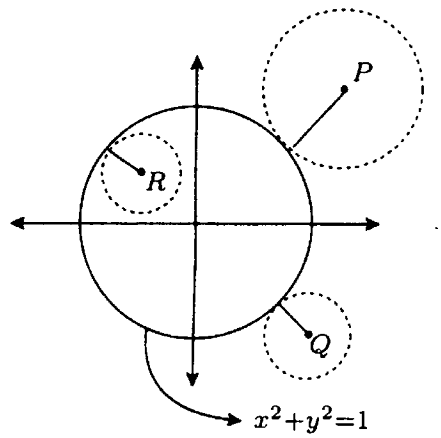

Figure 1.15: $\{x^2 + y^2 \neq 1\}$ is open

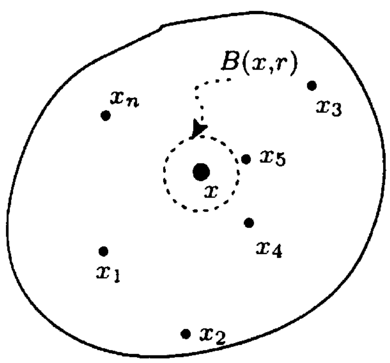

Figure 1.16: Space without finite points is open

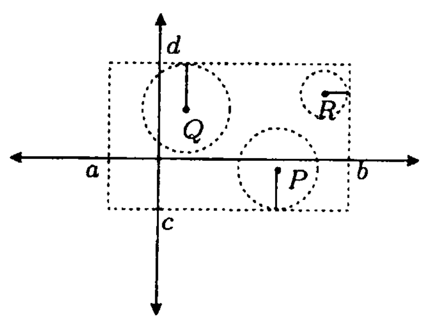

Figure 1.17: $(a, b) \times (c, d)$ is open in $\mathbb{R}^2$

Ex. 1.2.29. Let $U := \{(x, y) \in \mathbb{R}^2 : x > 0 \;\&\; y > 0\}$. Draw the figure of this subset. Is this open? Prove your claim.

Ex. 1.2.30. Let $U := \{(x,y) \in \mathbb{R}^2 : x \notin \mathbb{Z}, y \notin \mathbb{Z}\}$. Is U open in $\mathbb{R}^2$? Substantiate your claim.

Ex. 1.2.31. Let $U := \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \neq 1\}$. Show that U is open in $\mathbb{R}^2$. (Think geometrically and then make your ideas rigorous. See Figure 1.15.)

Ex. 1.2.32. Let A be any finite set in a metric space (X, d). Show that $X \setminus A$ is open. (See Figure 1.16.)

Ex. 1.2.33. What are all the open sets in a finite metric space?

Ex. 1.2.34. Show that a rectangle of the form $(a,b) \times (c,d)$ is open in $\mathbb{R}^2$. (See Figure 1.17.)

Ex. 1.2.35. Is a nonempty finite subset of $\mathbb{R}$ open? When is a singleton set $\{x\}$ in a metric space (X, d) open?

Ex. 1.2.36. Is $\mathbb{Q}$ open in $\mathbb{R}$? How about the set of irrational numbers?

Throughout the book, we need to deal with intervals in $\mathbb{R}$. A definition of intervals as special subsets of $\mathbb{R}$, rather than the list of all such subsets, will make our treatment rigorous.

Definition 1.2.37. We say that a subset $J \subset \mathbb{R}$ is an interval iff for every $x, y \in J$, and for every z such that $x < z < y$, it follows that $z \in J$.

The next proposition lists all intervals in $\mathbb{R}$.

Proposition 1.2.38. *Any subset $J \subset \mathbb{R}$ is an interval iff it is of the form (a,b), $[a,b]$, $[a,b)$, $(a,b]$, $(-\infty, \beta)$, (α, ∞), $[\alpha, \infty)$, $(-\infty, \beta]$ or $\mathbb{R}$. Here $a \leq b$ (Note that the empty set is an interval!) and $(\alpha, \infty) := \{x \in \mathbb{R} : x > a\}$ etc.*

Proof. If J is one of the types specified in the proposition, it is clearly an interval. So we need to prove the converse.

Let us break into two cases: (1) J is bounded and (2) J is unbounded.

Case (1): If $J = \emptyset$, there is nothing to prove. So we assume that J is nonempty.

Let $a := \inf J$ and $b := \sup J$. We claim that $(a,b) \subset J \subset [a,b]$. For, if $a < x < b$, then $x > a$ and since a is the greatest lower bound of J, x is not a lower bound for J. Hence there exists $c \in J$ such that $c < x$. Similarly using the fact that $b := \sup J$ and $x < b$, we find that there exists $d \in J$ such that $x < d$. Thus we have $c, d \in J$ such that $c < x < d$. Since J is an interval, it follows that $x \in J$. Since $x \in (a,b)$ is arbitrary, we have shown that $(a,b) \subset J$. By the very definitions of a and b, if $z \in J$, then $a \leq z \leq b$ and hence $J \subset [a, b]$. Now depending upon whether or not $a, b \in J$, we find that J must be one among $(a, b), (a, b], [a, b), [a, b]$.

We are sure that the reader should be able to supply the proof of Case (2) on his own and urge to reader to prove Case (2).

Case (2). Since J is unbounded there are three possibilities: (i) It is bounded above but not below, (ii) It is bounded below but not above and (iii) it is neither bounded above nor below. Let us look at (i). Let $b := \sup J$. We claim that $(-\infty, b) \subset J \subset (-\infty, b]$. Let $x \in (-\infty, b)$. Since J is not bounded below, x cannot be a lower bound for J. Therefore there exists $c \in J$ such that $c < x$. Since $x < b$ and b is the least upper bound for J, there exists $d \in J$ such that $x < d$. Thus, $c < x < d$ and hence $x \in J$. It is time that the reader went ahead on his own. $\square$

Ex. 1.2.39. Show that a (nonempty) subset of $\mathbb{R}$ is an interval iff it is convex.

Lemma 1.2.40 (Structure of open sets in $\mathbb{R}$). *A nonempty open set in $\mathbb{R}$ is the union of countable family of pairwise disjoint open intervals.*

Proof. We give a sketch of the proof. The details should be worked out by the reader.

Let $U \subset \mathbb{R}$ be open and $x \in U$. There exists an open interval J such that $x \in J \subset U$. Let J_x denote the union of all open intervals that contain x and contained in U. Then J_x is open. It is an interval since x is common to all the intervals that constitute J_x. (Why?) Given $x, y \in U$, we show that J_x and J_y are either identical or disjoint. Define an equivalence relation on U saying that $x \sim y$ iff $J_x = J_y$. Using this, we can write U as the union of a family, say, $\{J_i : i \in I\}$ of pairwise disjoint open intervals.

Choose one rational number $r_i \in J_i$ from each interval of this family. Then for $i \neq j$, $i, j \in I$, $r_i \neq r_j$ since $J_i \cap J_j = \emptyset$. We thus have a one-one map $i \mapsto r_i$ from I into $\mathbb{Q}$. Thus, I is countable. $\square$

Ex. 1.2.41. Let G be an open subset of $\mathbb{R}$ which is also a subgroup of the group $(\mathbb{R}, +)$. Show that $G = \mathbb{R}$. *Hint:* $0 \in G$ and hence $(-\varepsilon, \varepsilon) \subset G$ for some $\varepsilon > 0$. Use the fact that G is closed under addition.

Ex. 1.2.42. Let $\{U_i : i \in I\}$ be a family of open sets in a metric space (X, d). Show that the union $\cup_{i \in I} U_i$ is open in X.

Ex. 1.2.43. Let U_i, $1 \leq i \leq n$, be a finite collection of open sets in (X, d). Show that $\cap_{i=1}^n U_i$ is open in X.

Ex. 1.2.44. Show by means of an example that the intersection of an arbitrary family open sets need not be open. *Hint:* Can you think of $\{0\}$ as the intersection of a countable family of open intervals?

Ex. 1.2.45. What are the metric spaces in which the only open sets are $\emptyset$ and the full set? (Compare this with Ex. 1.2.22.)

Ex. 1.2.46. Show that a set $U \subset X$ of a metric space is open iff it is the union of open balls. *Hint:* Think of a family of open balls indexed by $x \in U$. See Section 4.1.

Ex. 1.2.47. If $(X, \| \ \|)$ is an NLS, U is open in X, then $x + U$ is open for any $x \in X$.

Ex. 1.2.48. If $(X, \| \ \|)$ is an NLS, U is open in X, then $A + U$ is open for any set $A \subset X$.

Ex. 1.2.49. Let $(X, \| \ \|)$ be an NLS. Show that if any vector subspace Y of X is open, then $Y = X$. *Hint:* Observe that $nB(0,r) = B(0,nr)$ and that $\cup_{n \in \mathbb{N}} B(0,nr) = X$.

Can you generalize this?

Ex. 1.2.50. Show that $C[0,1]$ is not an open subset of $(B[0,1], \| \ \|_\infty)$.

Theorem 1.2.51. *Let (X,d) be a metric space. Let $\mathcal{T}$ denote the set of all open sets in X. Then $\mathcal{T}$ has the following properties:*

(i) $\emptyset, X \in \mathcal{T}$.

(ii) *Arbitrary union of members of $\mathcal{T}$ again lies in $\mathcal{T}$.*

(iii) *The intersection of any finite number of members of $\mathcal{T}$ lies again in $\mathcal{T}$.*

$\mathcal{T}$ is called the topology determined by d.

Proof. The proof is simple and was broken into simple exercises above. The reader should prove the theorem on his own.

Proof of (i). Given any $x \in X$ and $r > 0$, by definition, $B(x,r) \subset X$. Hence X is open. I claim that $\emptyset$ is open. Suppose you challenge me. Then I ask you to give me a point x in it and I promise that I shall give you an $r > 0$ as required. You cannot and so I win! (If you are at present uncomfortable with this kind of argument, then we declare $\emptyset$ to be open. No proof required.)

Proof of (ii). Let $\{U_i : i \in I\}$ be a family of open sets. Let $x \in \cup_{i \in I} U_i$. Then $x \in U_j$ for some $j \in I$. Since U_j is open, there exists $r > 0$ such that $B(x,r) \subset U_j \subset \cup_{i \in I} U_i$. Since x is arbitrary, it follows that $\cup_{i \in I} U_i$ is open in X.

Proof of (iii). Let U_k, $1 \le k \le n$, be a finite collection of open sets in X. We are required to show that their intersection $\cap_{k=1}^n U_k$ is open.

If $\cap_{k=1}^n U_k = \emptyset$, then the intersection is open. If not, let $x \in \cap_{k=1}^n U_k$ be arbitrary. Since U_k is open and $x \in U_k$, there exists $r_k > 0$ such that $B(x, r_k) \subset U_k$. It is clear what we should do now. Let $r := \min\{r_k : 1 \leq k \leq n\}$. Then clearly, $B(x,r) \subset B(x, r_k) \subset U_k$ for each k, that is, $B(x,r) \subset \cap_{k=1}^n U_k$. Hence (iii) follows. □

Definition 1.2.52. Let X be a nonempty set. Assume that we are given a collection $\mathcal{T}$ of subsets of X satisfying the above properties (i)–(iii). Then $\mathcal{T}$ is called a *topology* on X and members of $\mathcal{T}$ are called the open sets in the topology $\mathcal{T}$. The pair $(X, \mathcal{T})$ is called a *topological space.*

Ex. 1.2.53. Show that on any nonempty set X there always exist topologies, at least two, provided X has more than one element.

Ex. 1.2.54. Let (X, d) be a metric space. Show that Hausdorff property of X is equivalent to saying that given two distinct points x, y, there exist open sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. (This allows us to define Hausdorff property of a topological space. See Definition 2.1.5.)

Ex. 1.2.55. Is the set U in Ex. 1.2.16 open?

Ex. 1.2.56. Let $X := C[0,1]$ with the sup norm metric $\| \; \|_\infty$. Let E be the set of all functions in X that do not vanish (that is, they do not take the value 0) at $t = 0$. Is E open in X?

Ex. 1.2.57. Show that the open ball $B_1(0,1) := \{f \in C[0,1] : \|f\|_1 < 1\}$ in $(C[0,1], \| \; \|_1)$ is open in $(C[0,1], \| \; \|_\infty)$. *Hint:* Observe that $\|f\|_1 \leq \|f\|_\infty$ for any $f \in C[0,1]$.

Ex. 1.2.58. Show that the open ball $B_\infty(0,1) := \{f \in C[0,1] : \|f\|_\infty < 1\}$ in the NLS $(C[0,1], \| \; \|_\infty)$, is not open in $(C[0,1], \| \; \|_1)$. *Hint:* For a positive function f, the norm $\|f\|_1$ admits a geometric interpretation. To be very explicit, we can construct positive continuous functions f on $[0,1]$ such that the area under the graph is less than any positive δ but whose maximum could be as large as we please. This implies that for no $\delta > 0$, the open ball $B_1(0, \delta)$ could be a subset of $B_\infty(0,1)$.

Ex. 1.2.59. Let X, Y be metric spaces. Consider the product set $X \times Y$ with the product metric (Ex. 1.1.32.) Show that any set of the form $B(x,r) \times B(y,s) \subset X \times Y$ is open in $X \times Y$. *Hint:* Let $(a,b) \in B(x,r) \times B(y,s)$. You need to find $\varepsilon > 0$ such that $B((a,b), \varepsilon) \subset B(x,r) \times B(y,s)$. See Figure 1.18. Work backwards. Any $\varepsilon < \min\{r - d(a,x), s - d(b,y)\}$ does the job.

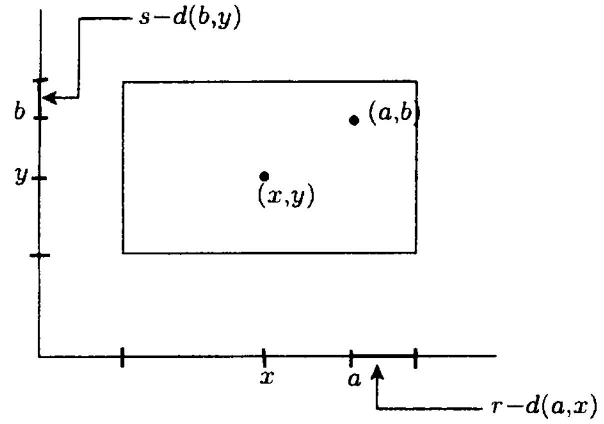

Figure 1.18: $B(x, r) \times B(y, s)$ is open in $X \times Y$

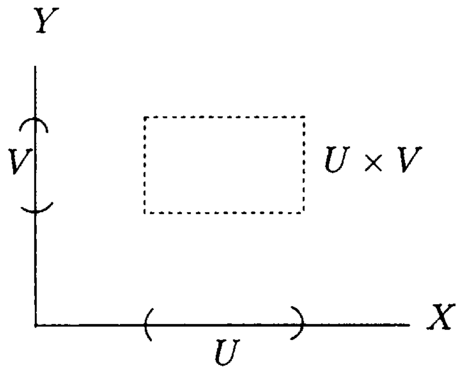

Figure 1.19: $U \times V$ is open in $X \times Y$

Ex. 1.2.60. Let X, Y be metric spaces. Consider the product set $X \times Y$ with the product metric (Ex. 1.1.32.) Let U (respectively, V) be an open set in X (respectively, Y). Show that $U \times V$ is open in $X \times Y$. (See Figure 1.19.)

Ex. 1.2.61. Keep the notations above. Let $W \subset X \times Y$ be open in the product metric. Let p_X and p_Y denote the projections of $X \times Y$ onto X and Y respectively. Show that $p_X(W)$ (respectively, $p_Y(W)$) is open in X (respectively, in Y). (See Fig 1.20.)

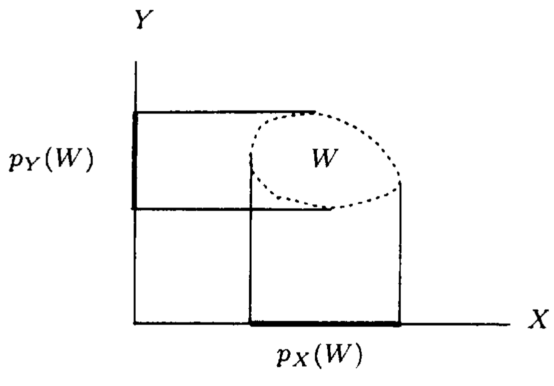

Figure 1.20: Projections of W on X and Y

Ex. 1.2.62. Let (X, d) be a metric space. Define δ on X as in Ex. 1.1.30. Show that a subset U is d-open iff it is δ-open.

Ex. 1.2.63. Let X be the set of sequences (x_k) such that $0 \leq x_k \leq 1/k$ for all $k \in \mathbb{N}$. Define $d(x, y) := (\sum_{k=1}^{\infty} |x_k - y_k|^2)^{1/2}$. Show that d is a metric on X. The metric space (X, d) is called the *Hilbert cube* and is usually denoted by I^ω or I^∞. *Hint:* Note that $\sum_{k=1}^{\infty} x_k^2 < \infty$ for any $x \in I^\infty$. Also, recall the triangle inequality in $\mathbb{R}^n$ for any n.

Ex. 1.2.64. Any open set in the Hilbert cube I^∞ (Ex. 1.2.63) is the union of open subsets of the form

$$U_1 \times \cdots U_n \times X_{n+1} \times \cdots \times X_{n+k} \times \cdots,$$

where $X_n := [0, 1/n]$ for $n \in \mathbb{N}$ and where U_i are open in X_i, $1 \leq i \leq n$ and n may vary.

Hint: Let U be open in I^∞ and $x \in U$. Let $\varepsilon > 0$ be such that $B(x, \varepsilon) \subset U$. Choose $n \in \mathbb{N}$ such that $\sum_{n+1}^{\infty}(1/k)^2 < \varepsilon^2/2$. Then the set

$$B(x_1, \frac{\varepsilon}{\sqrt{2n}}) \times \cdots \times B(x_n, \frac{\varepsilon}{\sqrt{2n}}) \times X_{n+1} \times \cdots \times X_{n+k} \times \cdots$$

is contained in U.

Ex. 1.2.65. Let d be the Euclidean metric on $\mathbb{R}^2$. Define

$$\delta(p, q) := \begin{cases} d(p, 0) + d(q, 0), & p \neq q \\ 0, & p = q, \end{cases}$$

for $p, q \in \mathbb{R}^2$. Show that δ is a metric on $\mathbb{R}^2$. What are $B_\delta(p, \varepsilon)$ for "sufficiently small" $\varepsilon > 0$ and $p \neq 0$? What are $B_\delta(0, \varepsilon)$ for $\varepsilon > 0$? Can you describe the δ-open sets?

Definition 1.2.66 (Equivalent Metrics). We say that two metrics d and d' on a set are *equivalent* if the topologies generated by d and d' are the same. This is equivalent to saying that $U \subset X$ is d-open iff it is d'-open.

Ex. 1.2.67. Two metrics d and d' on X are equivalent iff given any $x \in X$, any open ball $B_d(x, r)$ contains a ball $B_{d'}(x, \rho)$ for some $\rho > 0$ and any open ball $B_{d'}(x, s)$ contains $B_d(x, \sigma)$ for some $\sigma > 0$.

Ex. 1.2.68. What do Ex. 1.2.57 and Ex. 1.2.58 say about the metrics d_1 and d_∞ on $C([0, 1])$?

Ex. 1.2.69. We say that two norms $\| \; \|_1$ and $\| \; \|_2$ on a vector space V are *equivalent* if the metrics induced by them are equivalent. Show that the norms are equivalent iff there exist positive constants C_1, C_2 such that

$$C_1 \|x\|_1 \leq \|x\|_2 \leq C_2 \|x\|_1, \qquad (x \in V).$$

Hint: Ex. 1.2.67 applied to balls centred at 0 and Ex. 1.2.12.

Ex. 1.2.70. Show that the norms $\| \; \|_1$, $\| \; \|_2$ and $\| \; \|_\infty$ on $\mathbb{R}^n$ are equivalent. More precisely, show that for any $x \in \mathbb{R}^n$, we have

$$\frac{1}{n} \|x\|_1 \leq \frac{1}{\sqrt{n}} \|x\|_2 \leq \|x\|_\infty \leq \|x\|_2 \leq \|x\|_1 .$$

(In fact, all norms on $\mathbb{R}^n$ are equivalent, as we shall see later in Theorem 4.3.24.)

Ex. 1.2.71. Show that the norms $\| \ \|_1$ and $\| \ \|_\infty$ are not equivalent on $C[0,1]$. (See also Ex. 1.2.58 and Ex. 1.2.68.)

Example 1.2.72. Let (X,d) be any metric space. We claim that there exists a bounded metric on X which is equivalent to d.

One such is given by $\delta(x,y) := \min\{d(x,y), 1\}$ for $x, y \in X$. We verify only the triangle inequality, since other properties of a metric are obviously satisfied for δ.

Let $x, y, z \in X$. If $d(x,y) \leq 1$ and $d(y,z) \leq 1$, then observe that $\delta(x,y) = d(x,y)$ and $\delta(y,z) = d(y,z)$. Hence

$$\delta(x,z) \leq d(x,z) \leq d(x,y) + d(y,z) = \delta(x,y) + \delta(y,z).$$

We now assume that $d(x,y) > 1$. Then

$$\delta(x,z) \leq 1 \leq 1 + \delta(y,z) = \delta(x,y) + \delta(y,z).$$

If $d(y,z) > 1$, we proceed in a similar fashion.

Obviously, the metric δ is bounded, that is, X is bounded in δ. For, $X \subset B_\delta(x,r) := \{x' \in X : \delta(x',x) < r\}$ for any $x \in X$ and $r > 1$.

Let U be d-open and $x \in U$. Then there exists $r > 0$ such that $B_d(x,r) \subset U$. If this holds for r, then it holds for any $0 < \varepsilon < r$ so we may assume that $0 < r < 1$. In such a case, $B_d(x,r) = B_\delta(x,r)$ so that $B_\delta(x,r) \subset U$. Therefore, U is δ-open. Exactly similar argument shows that if V is δ-open, then it is d-open.

Ex. 1.2.73. Let (X,d) be a metric space and Y a set. Assume that $\varphi \colon X \to Y$ is a bijection. The we can *transfer* or *transport* the metric on X to Y in an obvious way:

$$\rho(y_1, y_2) := d(x_1, x_2) \text{ where } y_1 = f(x_1), y_2 = f(x_2).$$

Note that ρ is well-defined on Y. A concrete example is $\varphi \colon [1,\infty) \to (0,1]$ given by $\varphi(x) = 1/x$. Thus we get a new metric on $(0,1]$ by setting $\rho(x,y) = \left|\frac{1}{x} - \frac{1}{y}\right|$. Show that ρ is equivalent to the standard metric on $(0,1]$.

Ex. 1.2.74. Let $f \colon [0,\infty) \to [0,\infty)$ is a continuous function with the following properties:

(a) $f(t) = 0$ iff $t = 0$.

(b) f is nondecreasing: $f(x) \leq f(y)$ if $0 \leq x \leq y$.

(c) f is subadditive: $f(x+y) \leq f(x) + f(y)$ for all positive x, y.

If d is a metric on a set X, then $f \circ d$ is also a metric on X. The metrics d and $f \circ d$ are equivalent. In fact, the identity map is uniformly continuous from one metric space to the other.

As a specific example, consider $f(t) = \frac{t}{1+t}$.

Ex. 1.2.75 (Extended Real Line). Consider the function $\varphi\colon \mathbb{R} \to (-1,1)$ given by $\varphi(x) = \frac{x}{1+|x|}$.

(a) Show that φ is a bijection with the inverse $\psi(y) := \frac{y}{1-|y|}$.

(b) Observe that $\lim_{x\to\infty} \varphi(x) = 1$ and $\lim_{x\to-\infty} \varphi(x) = -1$. (Thus, the lines $y = \pm 1$ are horizontal asymptotes of the graph of φ.)

(c) Let $\mathbb{R}_e := \mathbb{R} \cup \{\pm\infty\}$. We extend the map φ to $\mathbb{R}_e$ by setting $\varphi(\infty) = 1$ and $\varphi(-\infty) = -1$. Then φ is a bijection of $\mathbb{R}_e$ with $[-1,1]$.

(d) Use the bijection to define a metric on $\mathbb{R}_e$.

(e) Show that the metric on $\mathbb{R}_e$ induces a metric on $\mathbb{R}$ which is equivalent to the standard metric.

Most of what follows can be done in a setting more general than the metric spaces. To illustrate this, whenever possible, we indicate the concepts and the results that hold for a topological space.

Definition 1.2.76 (Interior of a set). Let $S \subset X$ be a subset of a metric space. We say that $x \in S$ is an *interior* point of S if there exists $r > 0$ such that $B(x,r) \subset S$. The set of interior points of S is denoted by S^0. See Figure 1.21.

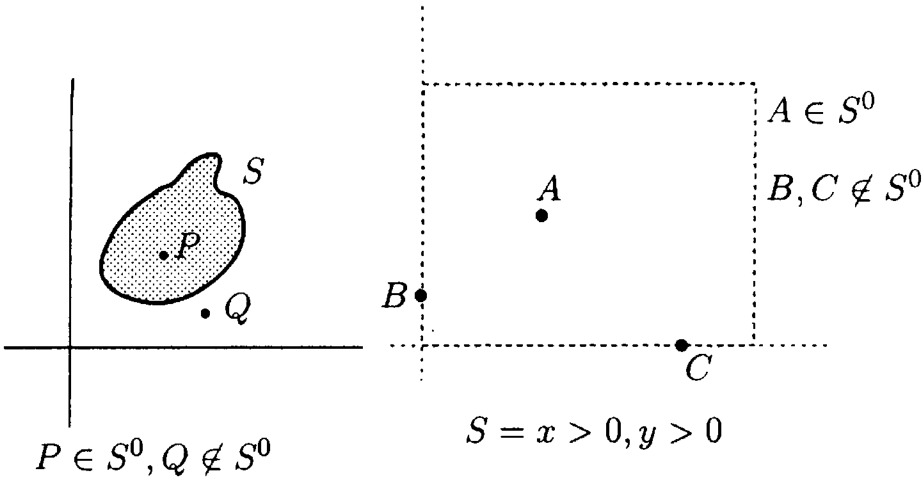

Figure 1.21: Interior of a set

If $(X, \mathcal{T})$ is a topological space, and $A \subset X$, a point $x \in A$ is said to be *an interior point* of A if there exists an open set $U \ni x$ such that $U \subset A$.

Thus, you may notice that the 'open ball' $B(x,r)$ has been replaced by an open set containing x in the definition.

If an exercise or a result holds for a topological space, then we formulate them using the following convention:

"Let X be a (metric) space. Then some result holds."

This means that the result is true for any topological space and you may assume that X is a metric space if you find it easier. See the next exercise.

Ex. 1.2.77. Let X be a (metric) space. Prove the following:
(a) A is open iff each of its points is an interior point, that is, A is open iff $A = A^0$.
(b) For any set A, the set A^0 is the largest open set contained in A.

Ex. 1.2.78. What is the interior of $\mathbb{Q}$ in $\mathbb{R}$? What is the interior of $(0,1] \subset \mathbb{R}$? What is the interior of the closed unit disk $B[0,1] \subset \mathbb{R}^2$?

Ex. 1.2.79. Let $A = [0,1) \subset \mathbb{R}$ have the induced metric from $\mathbb{R}$. Find $B_A(0,r)$ for any $r > 0$. Here $B_A(x,r)$ stands for the open ball *in* A centred at x and radius r with respect to the induced metric.

Ex. 1.2.80. Let $A = \{(x,y) : x \geq 0, y \geq 0\}$ be endowed with the induced metric as a subset of $\mathbb{R}^2$ with the Euclidean metric. Draw $B_{(A,d)}(\mathbf{0},1)$. (See Figure 1.22.)

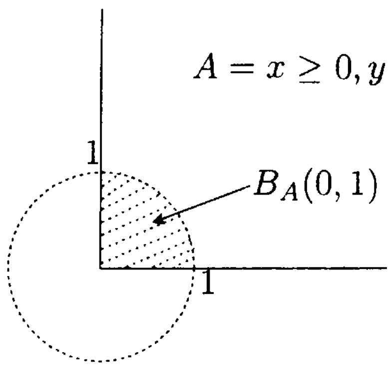

Figure 1.22: $B_A(0,1)$ is the shaded region

Ex. 1.2.81. Let (X,d) be a metric space. Let $A \subset X$ and $a \in A$. Let $B_A(a,r)$ denote the open ball in A with the induced metric. Give a description of $B_A(a,r)$. See Figure 1.23.

Definition 1.2.82 (Subspace Topology). Let Y be subset of a metric space. A subset $A \subset Y$ is said to be *open in* Y if it is an open subset of the metric space (Y,d) where d is the induced metric on Y.

Ex. 1.2.83. Show that $A \subset Y$ is open in Y iff there exists an open set U in X such that $A = Y \cap U$.

Ex. 1.2.84. Let $Y := \{(x,y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$ be the first quadrant in $\mathbb{R}^2$. Let $A := \{(x,y) \in Y : 0 \leq x < 1, 0 \leq y < 1\}$. Is A open in Y?

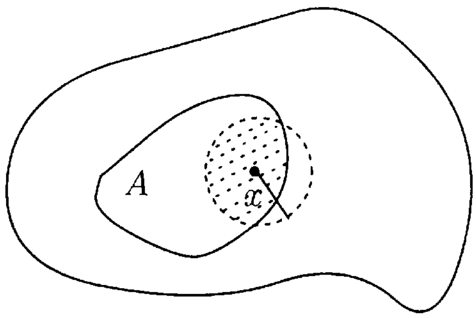

Figure 1.23: $B_A(x, r)$ is the shaded region

Ex. 1.2.85. Show that the collection $\mathcal{T}_Y$ of all sets open in Y is a topology on Y, called the *subspace topology.*

Now how do we define a subspace topology if Y is a subset of a topological space X?

This exercise answers the question posed in Definition 1.2.82.

Definition 1.2.86. Let $Y \subset X$ be a subset of a topological space X. The *subspace topology* $\mathcal{T}_Y$ is the collection $\{U \cap Y : U \in \mathcal{T}\}$.

Ex. 1.2.87. Let $X = \mathbb{R}$ and $Y = \mathbb{Z}$. Which subsets of $\mathbb{Z}$ are open in $\mathbb{Z}$?

Ex. 1.2.88. Let X be a (metric) space. Let $Y \subset X$ be open in X. Then $Z \subset Y$ is open in Y iff Z is open in X.

The result is not true if Y is not open in X.

Closed Sets

Definition 1.2.89. A subset $F \subset X$ of a (metric) space is said to be *closed* if its complement $X \setminus F$ is open in X.

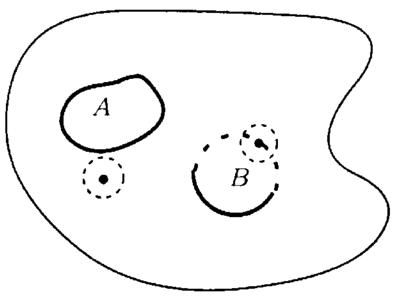

Figure 1.24: A is closed but B is not

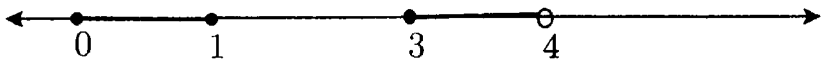

Figure 1.25: $[0,1]$ is closed but not [3,4)

Example 1.2.90. Consider two sets $A := [0,1]$ and $B := [3,4)$ in $\mathbb{R}$. See Figure 1.25.

We claim that A is closed while B is not. Let $x \notin A$. Then either $x < 0$ or $x > 1$. Let $r := |x|$ or $r = x - 1$. Then $(x - r, x + r) \subset A^c$. Let us now consider B. The point $x = 4 \notin B$. But any open 'ball' centred at 4 is of the form $(4 - \varepsilon, 4 + \varepsilon)$ which does intersect B. Hence the complement $\mathbb{R} \setminus B$ is not open and hence B is not closed.

Ex. 1.2.91. Show that $\emptyset$ and X are both open and closed in any (metric) space.

Ex. 1.2.92. Any finite subset of a metric space is closed.

Ex. 1.2.93. Let (X, d) be a discrete metric space. Find all closed sets in X.

Ex. 1.2.94. Let $X = \mathbb{R}^2$ with the standard metric. Let A be the union of the x and y-axes, that is the set $\{(x,y) \in \mathbb{R}^2 : xy = 0\}$. Show that A is closed.

Ex. 1.2.95. Let $S^1 := \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ be the unit circle in $\mathbb{R}^2$. Show that S^1 is closed.

Ex. 1.2.96. Is the set $\mathbb{Q}$ of rationals closed in $\mathbb{R}$? How about the set of irrationals?

Ex. 1.2.97. Show that the set $(0,1]$ is neither closed nor open in $\mathbb{R}$. *Hint:* Which points do you think will give rise to problem when we try to prove that the set is open or its complement is open?

Ex. 1.2.98. Give at least three 'distinct' subsets of $\mathbb{R}^2$ which are neither open nor closed.

Ex. 1.2.99. Let X be a (metric) space. Show that arbitrary intersections of closed sets is closed and a finite union of closed sets is closed. Find "counterexamples" to obvious generalizations.

Ex. 1.2.100. Let E be the xy-plane in $\mathbb{R}^3$. Is it closed? More generally, if P is any plane, say, given by $ax + by + cz = d$, is it closed? (Geometric thinking is encouraged.)

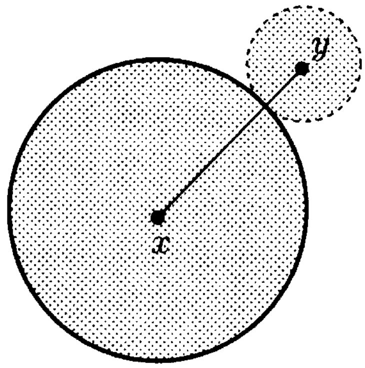

Figure 1.26: $B[x, r]$ is closed

Ex. 1.2.101. Show that any closed ball $B[x, r]$ in a metric space (X, d) is a closed set. (See Figure 1.26.)

Ex. 1.2.102. Let $a \in X$ be a point in the metric space X and $r > 0$. Show that the set $\{x \in X : d(x, a) = r\}$ is closed in X.

Ex. 1.2.103. Let Δ denote the sides along with the "inside" of the triangle whose vertices are at $(-1, 0)$, $(1, 0)$ and $(0, 1)$. Show that Δ is closed.

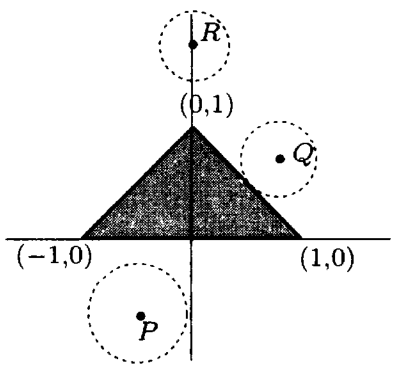

Figure 1.27: Δ is closed

Ex. 1.2.104. If F is a closed subset $\mathbb{R}^n$ and $x \in \mathbb{R}^n$, is $x+F$ still closed? Can you generalize this question?

Ex. 1.2.105. Show that in an NLS X, if F is closed and λ is scalar, then λF is closed.

Ex. 1.2.106. Show that there exist closed sets F and C in $\mathbb{R}^n$ such that their sum $F + C$ is not closed. (Think geometrically in $n = 2$. Think of a curve with two parallel lines as asymptotes. Or, let $F := \{(x, y) \in \mathbb{R}^2 : xy \geq 1\}$ be the set of points (x, y) which lie on and above the arm of the hyperbola $xy = 1$ in the first quadrant and C be the y-axis. Then $F + C = \{(x, y) : x > 0\}$. (See Figures 1.28–1.29.) Can you also think of such an example using the tan function in $(-\pi/2, \pi/2)$? See also Ex. 2.5.7.)

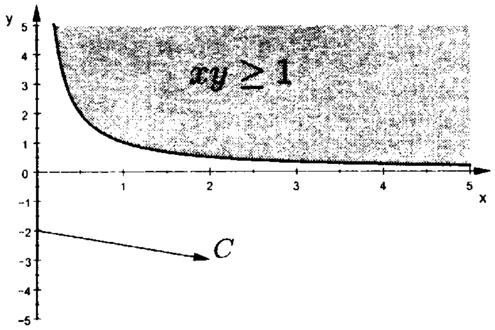

Figure 1.28: F and C in Ex. 1.2.106

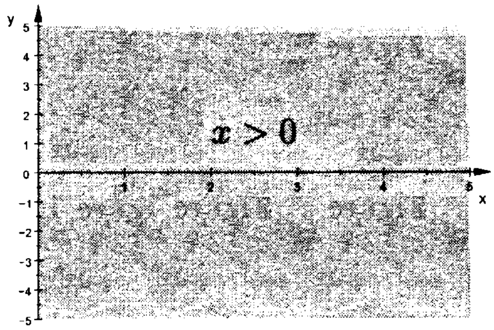

Figure 1.29: $F + C$ in the Ex. 1.2.106

Ex. 1.2.107. Since closed sets are defined as complements of open sets, it should be easy to find the counterpart of Theorem 1.2.51 (page 24) for the class of closed sets. Think over this and formulate such a result.

The following is the counterpart of Theorem 1.2.51 for the class of closed sets.

Theorem 1.2.108. *Let $\mathcal{C}$ denote the class of closed sets in a (metric) space X. Then*

(i) $\emptyset, X \in \mathcal{C}$.

(ii) *The intersection of an arbitrary family of closed sets is closed.*

(iii) *The union of a finite family of closed sets is closed.*

Proof. The idea is to 'take complements' in the proof of Theorem 1.2.51 (page 24).

For instance, we shall prove (ii). Let $\{F_i : i \in I\}$ be a family of closed sets. Since F_i is closed, its complement $U_i := X \setminus F_i$ is open in X for each $i \in I$. Hence, their union $\cup_{i \in I} U_i$ is open and hence its complement

$$(\cup_{i \in I} U_i)^c = \cap_{i \in I} U_i^c = \cap_{i \in I} F_i$$

is closed. □

Ex. 1.2.109. Let A be a nonempty subset of a metric space (X, d). Show that $B \subset A$ is closed *in* A iff there exists a closed set F of X such that $B = F \cap A$. (Compare this with Ex. 1.2.83.)

Ex. 1.2.110. Show that $\{x \in \mathbb{Q} : -1 < x < 1\}$ is open in $\mathbb{Q}$ but not closed in $\mathbb{Q}$ and that $\{x \in \mathbb{Q} : -\sqrt{2} < x < \sqrt{2}\}$ is both open and closed in $\mathbb{Q}$.

Chapter 2

Convergence

2.1 Convergent Sequences

Definition 2.1.1. A *sequence* in a (metric) space X is a function $x\colon \mathbb{N} \to X$. We exhibit the sequence x as (x_n) where $x_n := x(n)$.

Given a sequence x in a metric space, a subsequence is the restriction of x to an infinite subset $S \subset \mathbb{N}$. If we exhibit S as $n_1 < n_2 < \cdots < n_k < \cdots$, then we write the subsequence (x_{n_k}). An easy but important and useful observation about the indices n_k of the subsequence (x_{n_k}) is that $n_k \geq k$ for all k. For, $n_1 \geq 1$ and since $n_2 > n_1$, it follows that $n_2 \geq 2$. By induction, assume that $n_k \geq k$. Since $n_{k+1} > n_k \geq k$, it follows that $n_{k+1} \geq k + 1$.

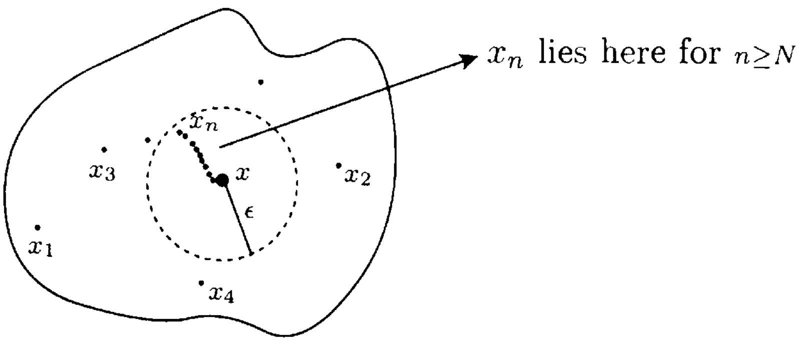

Figure 2.1: Convergence of a sequence

We say that a sequence (x_n) *converges* to $x \in X$ if given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $x_n \in B(x, \varepsilon)$. See Figure 2.1. This is same as saying that for all $n \geq N$, $d(x, x_n) < \varepsilon$. We then say that the sequence is *convergent*. The point x is called the *limit* of the sequence and denote it by $x = \lim_{n\to\infty} x_n$ and also by $x_n \to x$.

How do we define a convergent sequence in an arbitrary topological space X? We say that the sequence (x_n) in X converges to a point $x \in X$ if for any given open set $U \ni x$,[1] there exists N (which may depend on U) such that $x_n \in U$ for $n \geq N$.

Ex. 2.1.2. Show that any subsequence of a convergent sequence converges to the limit of the given sequence.

Lemma 2.1.3. *The limit of a sequence in a metric space is unique.*

Proof. What we are required to prove is this. If (x_n) is a sequence in X such that $x_n \to x$ and $x_n \to y$, then $x = y$.

Strategy: Assume $x \neq y$. Look at Figure 2.2 and use the Hausdorff property of X. Can we find an $r > 0$ such that $B(x,r)$ and $B(y,r)$ are disjoint? If possible, then for all large values of n, the terms x_n must lie in both the balls. That will be a required contradiction.

If $x \neq y$, then $\delta := d(x,y) > 0$. We choose $r \leq \delta/2$ and consider the open balls $B(x,r)$ and $B(y,r)$. We claim that they are disjoint. For, if $z \in B(x,r) \cap B(y,r)$, then

$$\delta = d(x,y) \leq d(x,z) + d(z,y) < r + r \leq \delta,$$

a contradiction. Now, since $x_n \to x$, there exists n_1 such that $x_n \in B(x,r)$ for all $n \geq n_1$. Similarly, there exists n_2 such that $x_n \in B(y,r)$ for $n \geq n_2$. In particular, for all $n \geq \max\{n_1, n_2\}$, we see that $x_n \in B(x,r) \cap B(y,r)$. This contradicts the fact that the balls are disjoint. So we conclude that $x = y$.

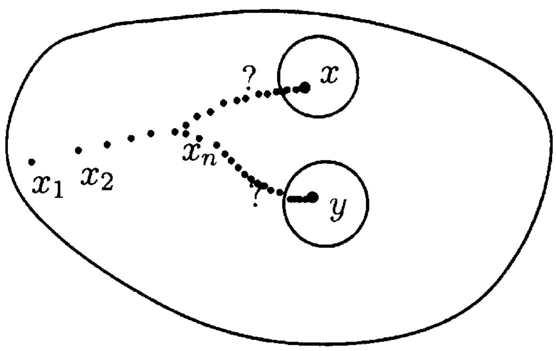

Figure 2.2: Uniqueness of limit: Where will x_n's go?

A standard proof runs as follows. Let $\varepsilon > 0$ be given. Then there exists n_1 and n_2 as above. Now if we choose $n \geq \max\{n_1, n_2\}$, then, we have

$$d(x,y) \leq d(x,x_n) + d(x_n,y) < 2\varepsilon.$$

[1] $U \ni X$ is same as saying that $x \in U$.

Since ε is arbitrary, we conclude that $d(x, y) = 0$ and hence $x = y$.

Why did we do the first proof? Because, it is more geometric and generalizes to a wider context. See the remark below. □

Remark 2.1.4. The last lemma is false in an arbitrary topological space. We do not stop to give an example of such a phenomenon. But we shall introduce a special class of topological spaces for which the lemma remains true.

Definition 2.1.5. We say that a topological space X is Hausdorff if for any pair of distinct points $x, y \in X$, we can find open sets $U \ni x, V \ni y$ such that $U \cap V = \emptyset$.

Refer to Ex. 1.2.18 and Ex. 1.2.54.

Ex. 2.1.6. Show that a sequence (x_n) in a metric space converges X to $x \in X$ iff the sequence $d(x_n, x)$ converges to .... Complete the sentence and prove it.

Ex. 2.1.7. Let $x_k = (x_{k1}, \dots, x_{kn}) \in \mathbb{R}^n$. Show that (x_k) converges to $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ iff $x_{ki} \to x_i$ as $k \to \infty$ for each i. *Hint:* An important observation is $|x_j| \le \|x\|$ $(1 \le j \le n)$ for any $x = (x_1, \dots, x_n) \in \mathbb{R}^n$.

Ex. 2.1.8. Let $x_k \in \mathbb{R}^n$ converge to $x \in \mathbb{R}^n$. Show that $\|x_k\| \to \|x\|$. Is the converse true?

Ex. 2.1.9. Let $x_k \to x$ and $y_k \to y$ in $\mathbb{R}^n$. Prove that $x_k + y_k \to x + y$ and that $\langle x_k, y_k \rangle \to \langle x, y \rangle$.

Ex. 2.1.10. Let X, Y be metric spaces. Let $X \times Y$ be endowed with the product metric. Show that a sequence $(x_n, y_n) \in X \times Y$ converges to $(x, y) \in X \times Y$ iff $x_n \to x$ in X and $y_n \to y$ in Y.

Example 2.1.11. What does it mean to say that a sequence (f_n) in $B(X)$ (with sup norm) is convergent with the induced metric $d_\infty(f, g) := \|f - g\|_\infty$?

Let $f_n \to f$ in the metric d_∞. This means that, given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \ge N$ we have $d_\infty(f_n, f) < \varepsilon$. Expanding this, we get $\sup\{|f_n(x) - f(x)| : x \in X\} < \varepsilon$ for $n \ge N$. In particular, $|f_n(x) - f(x)| < \varepsilon$ for all $x \in X$ and $n \ge N$. Thus, we conclude that if $f_n \to f$ in d_∞, then f_n converges to f uniformly on X.[2]

The converse is also true. That is, if $f_n \in B(X)$ converge to an $f \in B(X)$ uniformly on X, then $f_n \to f$ in the metric. For, if $\varepsilon > 0$ is given,

[2] We say that a sequence (f_n) of real (or complex) valued functions on a set E converge uniformly on E to a function f if for a given $\varepsilon > 0$, there exists n_0 such that for all $n \ge n_0$ we have $|f(x) - f_n(x)| < \varepsilon$ for all $x \in E$ and $n \ge n_0$.

by the uniform convergence, we can find N such that $|f_n(x) - f(x)| < \varepsilon/2$ for all $n \geq N$ and for all $x \in X$. Taking the supremum of the inequality over $x \in X$, we see that $\sup\{|f_n(x) - f(x)| : x \in X\} \leq \varepsilon/2 < \varepsilon$, that is, $\|f_n - f\|_\infty < \varepsilon$ for $n \geq N$. Thus, $f_n \to f$ in the metric.

In summary, a sequence f_n in $(B(X), \| \ \|_\infty)$ converges to $f \in B(X)$ iff f_n converges to f uniformly on $[0,1]$. For this reason, the sup norm is also called the uniform norm.

Ex. 2.1.12. When does a sequence (f_n) converge in $(C[0,1], \| \ \|_\infty)$?

Ex. 2.1.13. Let $f_n(x) := x^n$ for $x \in [0,1]$, for $n \in \mathbb{N}$. Show that the sequence (f_n) is convergent in $(C[0,1], \| \ \|_1)$ whereas it is not convergent in $(C[0,1], \| \ \|_\infty)$.

Ex. 2.1.14. Let (x_n) be a sequence in a discrete metric space. When does it converge? (Classify all convergent sequences in a discrete metric space.)

Let X be a nonempty set. If we let $\mathcal{T}$ to be the family of all subsets of X, then $\mathcal{T}$ is a topology on X, called the *discrete topology*. The space $(X, \mathcal{T})$ is then called a discrete space. Find all convergent sequences in a discrete space.

Ex. 2.1.15. Consider $M(n, \mathbb{R})$ as an NLS as in Ex. 1.1.34. Then a sequence (A_k) in $M(n, \mathbb{R})$ converges to $A \in M(n, \mathbb{R})$ iff the matrix entries $a_{ij}^k \to a_{ij}$ as $k \to \infty$ for all i, j.

Ex. 2.1.16. Let the notation be as above. Let $A_k \to A$. Then $A_k^2 \to A^2$. Can you generalize this?

Ex. 2.1.17. Let (A_k) be a sequence of invertible matrices in $M(n, \mathbb{R})$ converging to an $A \in M(n, \mathbb{R})$. Is it necessary that A is invertible?

Ex. 2.1.18. Let $A_k \in M(n, \mathbb{R})$ converge to $A \in M(n, \mathbb{R})$. Show that $\det(A_k) \to \det A$. *Hint:* Look at the case when $n = 2$.

Ex. 2.1.19 (p-adic Metrics). (a) Let a prime number p be fixed. Given a nonzero $x \in \mathbb{Q}$, we can write it as $x = p^k \frac{m}{n}$ where k, m, n are integers with p dividing neither m nor n. We define $v_p(x) = k$. We define the p-adic metric d_p as follows:

$$d_p(x,y) = \begin{cases} 0 & \text{if } x = y \\ p^{-v_p(x-y)} & \text{if } \neq y. \end{cases}$$

Show that $(\mathbb{Q}, d_p)$ is a metric space. *Hint:* d_p satisfies a stronger form of the triangle inequality:

$$d_p(x,z) \leq \max\{d_p(x,y), d_p(y,z)\}.$$

Show this by observing $v_p(a-b) \geq \min\{v_p(a), v_p(b)\}$ for $a, b \in \mathbb{Q}$, $a \neq 0$, $b \neq 0$ and $a \neq b$

(b) Let d be a metric on a set X. Assume that it satisfies the *ultra-metric* inequality:

$$d(x,z) \leq \max\{d(x,y), d(y,z)\}. \tag{2.1}$$

Examples of such metrics are the p-adic metrics d_p and the metric in Ex. 1.1.28. Prove the following:

(i) Equality holds in (2.1) whenever $d(x,y) \neq d(y,z)$.

(ii) Any ball in (X,d) is both closed and open.

(iii) Every point in a ball is the centre of the ball.

(c) Show the sequence (p^n) converges in $(\mathbb{Q}, d_p)$ but not in $(\mathbb{Q}, d)$ where d is the restriction to $\mathbb{Q}$ of the absolute value metric on $\mathbb{R}$.

(d) What does it mean to say that a sequence of integers converge to 0 in $\mathbb{Q}$ with the p-adic metric?

(e) Show that the sequence (x_n) where $x_1 = 3$, $x_2 = 33$, $x_3 = 333$ and so on is convergent to $-1/3$ in $(\mathbb{Q}, |\ |_5)$.

2.2 Limit and Cluster Points

Definition 2.2.1. Let X be a metric space. Let $E \subset X$. A point $x \in X$ is a *limit point* of E iff for every $r > 0$, we have $B(x,r) \cap E \neq \emptyset$. See Figure 2.3.

How do we extend the notion of a limit point to an arbitrary topological space? See Ex. 2.2.5 and Definition 2.2.6.

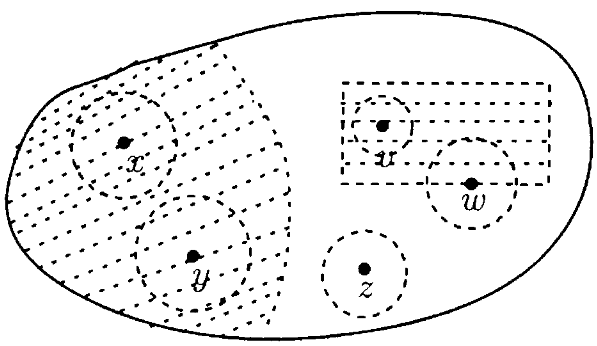

Figure 2.3: Limit Points

Ex. 2.2.2. Let E be a subset of a (metric) space. Show that any $x \in E$ is a limit point of E.

Ex. 2.2.3. Find the limit points of $(0,1)$, $(0,1]$ and $[0,1]$ in $\mathbb{R}$.

Ex. 2.2.4. Find the set of all limit points of $\mathbb{Q}$ in $\mathbb{R}$.

Ex. 2.2.5. Show that x is a limit point of E iff every open set containing x has nonempty intersection with E.

Definition 2.2.6. Let $E \subset X$ be a subset of a topological space X. A point $x \in X$ is said to be a limit point of E if every nonempty open set that contains x contains a point of E.

Ex. 2.2.7. Let X be a metric space and $E \subset X$. A point x is a limit point of E iff there exists a sequence (x_n) in E such that $x_n \to x$.

This result is false in an arbitrary topological space. It is true for a special class of topological spaces known as first countable T_1 spaces, which we do not intend to define! In fact, the reader should analyze the proof and arrive at the conditions to be satisfied by a topological space so that the results continues to be true.

The next theorem and its corollary offer an important method of proving that a set E of a metric space is closed.

Theorem 2.2.8. *A subset E of a metric space (X, d) is closed iff E contains all its limit points.*

Proof. This is an easy exercise. The reader should attempt a proof on his own.

Assume that E is closed. Let x be a limit point of E. If $x \in E$, there is nothing to prove. If $x \notin E$, then the set $U := X \setminus E$ is open and $x \in U$. Therefore there exists an $r > 0$ such that the open ball $B(x, r) \subset U$. Hence, $B(x, r) \cap E = \emptyset$. This contradicts the assumption that x is a limit point of E. Therefore we conclude that the statement $x \notin E$ is false. In other words, any limit point of E lies in E.

Conversely, if E contains all its limit points, then we want to prove that E is closed. Consider $U := X \setminus E$. We need to prove that U is open. If U is not open, it follows that there exists $x \in U$ such that for every $r > 0$, the open ball $B(x, r)$ is not contained in U. That is, for every $r > 0$, there exists a point in the complement of U. Hence x is such that for every $r > 0$, the set $B(x, r) \cap A \neq \emptyset$. That is, x is limit point of E and it is not in E. This contradicts our hypothesis that E contains all its limit points. Hence we conclude that U is open and hence E is closed. □

The following is an immediate corollary of the theorem and Ex. 2.2.7.

Corollary 2.2.9. *A subset E of a metric space (X, d) is closed iff the following holds: If (x_n) is a sequence in E converging to $x \in X$, then $x \in E$.* □

Example 2.2.10. Let $C := C[0,1]$ be the set of real valued continuous functions on $[0,1]$, considered as a subset of $B[0,1]$ under the norm $\| \ \|_\infty$. Then if $f \in B[0,1]$ is a limit point of C, then $f \in C$. Hence C is closed in $B[0,1]$ by Theorem 2.2.8. *Hint:* This is a well-known result from real analysis in disguise. See Theorem 6.1.7.

Ex. 2.2.11. Let $E \subset \mathbb{R}$ be a nonempty bounded and closed subset. Show that $\sup E$, $\inf E \in E$.

Ex. 2.2.12. Show that the only nonempty subset of $\mathbb{R}$ which is both open and closed in $\mathbb{R}$ is $\mathbb{R}$. *Hint:* If A is a nonempty subset of $\mathbb{R}$ which is both open and closed, let $x \in A$. Then $(x - \varepsilon, x + \varepsilon) \subset A$ for some $\varepsilon > 0$. Let $\beta := \sup\{b : (x - \varepsilon, b) \subset A, b > x\}$. Why cannot β be finite? If β were finite, conclude that $\beta \in A$.

Ex. 2.2.13. Let E be a subset of a metric space with the following property: If a sequence (x_n) *in* X converges to a point $x \in E$, then there exists $N \in \mathbb{N}$, such that $x_n \in E$ for all $n \geq N$. Show that E is open.

Ex. 2.2.14. Show that the diagonal $\{(x,x) : x \in X\}$ is closed in the product metric space $X \times X$.

Ex. 2.2.15. If A is a nonempty bounded subset of $\mathbb{R}$, then its supremum and infimum are limit points of A.

Definition 2.2.16 (Closure of a set). Let (X,d) be a metric space and $A \subset X$. Let $\lim(A)$ denote the set of all limit points of A. Thus

$$\lim(A) := \{y \in X : \text{ For every } r > 0, \text{ the set } B(y,r) \cap A \neq \emptyset\}.$$

Ex. 2.2.17. Show that $\lim(A)$ is the smallest closed set containing A. The standard notation for $\lim(A)$ is $\overline{A}$. It is usually called the *closure* of A.

The last exercise allows us to define the closure of a set in a topological space.

Definition 2.2.18. Let A be a subset of a topological space. Then the *closure* $\overline{A}$ of A in X is the smallest closed set that contains A. (Why does this make sense? How do we know that there exists the smallest closed set containing A?)

Ex. 2.2.19. Show that in a metric space $\overline{B(x,r)} \subset B[x,r]$. Give an example to show that $\overline{B(x,r)}$ can be a proper subset of $B[x,r]$.

Ex. 2.2.20. In $\mathbb{R}^n$, show that $\overline{B(x,r)} = B[x,r]$. Extend the result to normed linear spacess.

Ex. 2.2.21. Find the closures of the following subsets of $\mathbb{R}$: (a) $\mathbb{Q}$, (b) $\mathbb{Z}$, (c) $(-1,0) \cup \mathbb{N}$ and (d) $(0,1) \cup [3,4]$.

Ex. 2.2.22. Investigate the relation between the closures of the sets $A \cup B$, $A \cap B$, $A \dot{\subset} B$ and the sets $\overline{A}$, $\overline{B}$ in an arbitrary (metric) space.

Definition 2.2.23. We say that x is a *cluster* or an *accumulation* point of a set E if for each $r > 0$, the set $B(x,r) \cap E$ contains a point *other than* x.

If we let $B'(x,r) := B(x,r) \setminus \{x\}$, then $B'(x,r)$ is called a *deleted neighbourhood* of x. Thus, x is a cluster point of E iff every deleted neighbourhood of x contains a point of E.

Almost all text-books refer to these points also as limit points. However in this book, we shall distinguish between limit points and cluster points.

We shall give an analogy of cluster points in real life. If a celebrity, say, Amitabh Bachchan or Sachin Tendulkar attends a party in India, then in any vicinity around him, you will always find some other member of the party! If you or I were there in a party, it might happen that we may be sitting in a corner and in our vicinity there are no other members of the party. Thus, they are cluster points of the set of India people, though they may not be, say, among some tribal people in Africa!

Ex. 2.2.24. Show that every point $\mathbb{Z} \subset \mathbb{R}$ is limit point of $\mathbb{Z}$ while $\mathbb{Z}$ has no cluster point.

What are the cluster points of $\mathbb{Q}$ in $\mathbb{R}$?

Ex. 2.2.25. Let $A := \{1/n : n \in \mathbb{N}\} \subset \mathbb{R}$. Show that 0 is the only cluster point of A.

Ex. 2.2.26. Show that every point of a nonempty open set U in $\mathbb{R}^n$ is a cluster point of U.

Ex. 2.2.27. Let $A \subset \mathbb{R}^n$. Then $x \in \mathbb{R}^n$ is a cluster point of A iff every open set containing x contains infinitely many points of A.

Theorem 2.2.28 (Bolzano Weierstrass Theorem). *Let A be an infinite bounded subset of $\mathbb{R}$. Then there is a cluster point of A in $\mathbb{R}$.*

Proof. Let $E := \{x \in \mathbb{R} : x \leq a \text{ for infinitely many } a \in A\}$. Let $M \in \mathbb{R}$ be such that $-M \leq a \leq M$ for all $a \in A$. It is obvious that $-M \in E$. We can easily show that E is bounded by M. Hence there exists $\ell \in \mathbb{R}$ such that $\ell = \sup E$. We claim that ℓ is a cluster point of E. That is, we need to show that for any given $\varepsilon > 0$ there exists a point $a \in (\ell - \varepsilon, \ell + \varepsilon) \cap A$ other than ℓ itself.

Since $\ell - \varepsilon$ is not an upper bound for E there is an $x \in E$ such that $\ell - \varepsilon < x$. Since $x \in E$ there exist infinitely many elements $a \in A$ such that $x \leq a$. Hence there exist infinitely many elements $a \in A$ such that $\ell - \varepsilon < a$. Also for infinitely many such a we have $a < \ell + \varepsilon$. For, otherwise, except for finitely many such elements of A for all other $a \in A$ we have $a \geq \ell + \varepsilon$. But then $\ell + \varepsilon \in E$. This contradicts the fact that $\ell = \sup E$. Thus there exist infinitely many $a \in A$ such that $\ell - \varepsilon < a < \ell + \varepsilon$. In particular there is at least one $a \in A \cap (\ell - \varepsilon, \ell + \varepsilon)$ which is different from ℓ. □

Remark 2.2.29. The version of Bolzano-Weierstrass in terms of sequences is: *Any bounded sequence of reals has a convergent subsequence* follows from Theorem 2.2.28. If the image of the sequence is finite then there exists an $x \in \mathbb{R}$ such that $x = x_n$ for infinitely many $n \in \mathbb{N}$. These n's give rise to a subsequence which converges to x. If the image of the sequence is infinite then it is a bounded infinite subset of $\mathbb{R}$. Let x be a cluster point of this set. Let $x_{n_k} \in (x - 1/k, x + 1/k)$ be an element of the sequence chosen inductively so that $x_{n_{k+1}} \notin \{x_{n_1}, \ldots, x_{n_k}\}$. The subsequence (x_{n_k}) then converges to x.

Ex. 2.2.30. Extend the last two exercises to $\mathbb{R}^n$. Can one extend these to an arbitrary NLS?

Ex. 2.2.31. Let F be a finite subset of a (metric) space. What are its cluster points?

2.3 Cauchy Sequences and Completeness

Definition 2.3.1. A sequence (x_n) in a metric space is called a *Cauchy sequence* if for any given $\varepsilon > 0$, we can find $N \in \mathbb{N}$ such that whenever $m > N$ and $n > N$, we have $d(x_m, x_n) < \varepsilon$.

Ex. 2.3.2. Show that any convergent sequence in a metric space is Cauchy. The converse is not true. *Hint:* Let (x_n) converge to x. Given $\varepsilon > 0$, choose N such that $d(x_n, x) < \varepsilon$ for $n \geq N$. What can you say about $d(x_m, x_n)$ if $m, n \geq N$?

Ex. 2.3.3. What are the Cauchy sequences in a discrete metric space?

Example 2.3.4. Let X be any nonempty set. Let us consider the normed linear space $B(X)$ of all bounded real valued functions on X under the sup norm $\| \ \|_\infty$. We claim that a sequence $(f_n) \in B(X)$ is Cauchy iff it is uniformly Cauchy.

First of all, let us recall the definition of a uniformly Cauchy sequence of functions on X. We say that a sequence $f_n : X \to \mathbb{R}$ is *uniformly Cauchy* if for a given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ with the following property:

$$|f_n(x) - f_m(x)| < \varepsilon \text{ for all } x \in X \text{ and for all } m, n \geq N.$$

Let us assume that (f_n) is Cauchy in $B(X)$. Let $\varepsilon > 0$ be given. Since (f_n) is Cauchy in the NLS, there exists $N \in \mathbb{N}$ such that $\| f_m - f_n \|_\infty < \varepsilon$ for $m, n \geq N$. Unwinding the definition of the norm, we see that

$$\sup\{|f_m(x) - f_n(x)| : x \in X\} < \varepsilon, \text{ for } m, n \geq N.$$

In particular, for all $x \in X$, we have

$$|f_m(x) - f_n(x)| < \varepsilon, \text{ for } m, n \geq N. \tag{2.2}$$

Thus (f_n) is uniformly Cauchy on X. The proof of the converse is left to the reader. □

Ex. 2.3.5. Show that any Cauchy sequence in a metric space is bounded, that is, if (x_n) is Cauchy in a metric space (X, d), then x_n lies in an open ball $B(x, r)$ for all n. *Hint:* Apply the definition for $\varepsilon = 1$ to find N. Except for finitely many, all $x_n \in B(x_N, 1)$.

The next proposition lists some of the most often used facts about Cauchy sequences. It is in fact a compilation of the facts enunciated in the exercises above.

Proposition 2.3.6. *Let (X, d) be a metric space.*

1. *Any convergent sequence in (X, d) is Cauchy. The converse is not true.*
2. *Any Cauchy sequence (x_n) in X is bounded, that is, all $x_n \in B(x, r)$ for some $x \in X$ and $r > 0$.*
3. *A Cauchy sequence is convergent iff it has a convergent subsequence.*

Proof. You must have already proved the results. If not, go ahead and prove them on your own.

Proof of 1. Let (x_n) be Cauchy in X. Let $\varepsilon > 0$ be given. We need to find N such that if $m, n \geq N$ then $d(x_n, x_m) < \varepsilon$. We are given that (x_n) is convergent. Let $x_n \to x$. Draw an open ball $B(x, \varepsilon/2)$. For all n sufficiently large, $x_n \in B(x, \varepsilon/2)$. Hence for such m, n we expect that $d(x_m, x_n) < 2\varepsilon$. (See Figure 2.4.) We turn this idea into a proof. Let N be such that $d(x_n, x) < \varepsilon/2$. Then for $m, n \geq N$, we have

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

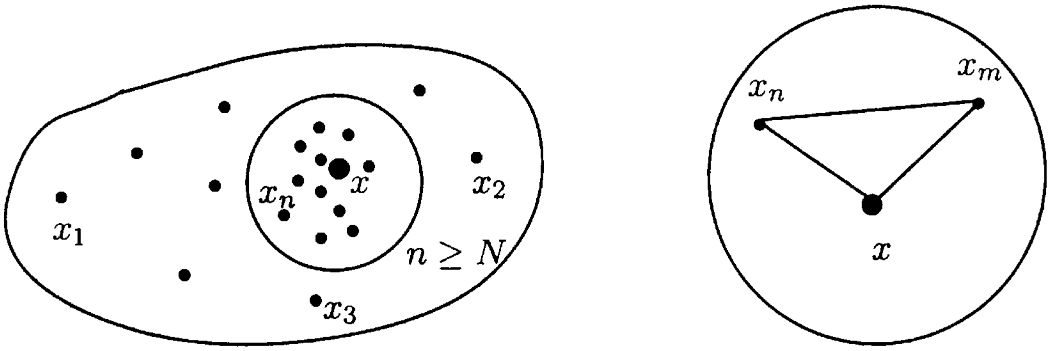

Figure 2.4: Convergent sequences are Cauchy

The converse is not true. For, if we take $(0,1)$ with the standard metric, then the sequence $(1/n)$ is Cauchy. Given $\varepsilon > 0$, choose $N > 1/\varepsilon$. Then, for all $n \geq m \geq N$,

$$|1/n - 1/m| = \left|\frac{n-m}{nm}\right| \leq \frac{n}{nm} = 1/m \leq 1/N < \varepsilon.$$

But the sequence does not converge to any point in $(0,1)$. For, if $x \in (0,1)$ is the limit of $(1/n)$, choose (using the Archimedean property of $\mathbb{R}$) a natural number $N \in \mathbb{N}$ such that $1/N < x$. We select $\varepsilon := x - 1/N > 0$. For this value of ε, the convergence of $1/n \to x$ implies that there exists n_0 such that for all $n \geq n_0$, we have $|x - 1/n| < \varepsilon$. If we choose $n \geq \max\{2N, n_0\}$, we find that

$$\left|x - \frac{1}{n}\right| \geq \left|x - \frac{1}{2N}\right| \geq \left|x - \frac{1}{N}\right|,$$

a contradiction. We therefore conclude that the Cauchy sequence $(1/n)$ is not convergent in $(0,1)$.

Proof of 2. Let us take $\varepsilon = 1$. Since (x_n) is Cauchy, there exists N such that $d(x_n, x_m) < 1$ for $n, m \geq N$. In particular, $x_n \in B(x_N, 1)$ for all $n \geq N$. Let

$$R > \max\{d(x_1, x_N), \ldots, d(x_{N-1}, x_N), 1\}.$$

Then $x_n \in B(x, R)$ for all $n \in \mathbb{N}$.

Proof of 3. If the given Cauchy sequence (x_n) is convergent, we can take it as the convergent subsequence. Let us prove the converse. Let (x_n) be Cauchy in X and assume that (x_{n_k}) is a convergent subsequence, converging to $x \in X$. We claim that $x_n \to x$.

Let $\varepsilon > 0$ be given. Choose N_1 such that $d(x_m, x_n) < \varepsilon/2$ for $m, n \geq N_1$. Choose k_1 such that if $k \geq k_1$, then $d(x, x_{n_k}) < \varepsilon/2$. Observe that

$n_k \geq k$. (For, $1 \leq n_1 < n_2$ implies $n_2 \geq 2$. By induction, we see that $n_k \geq k$.) Let $N := \max\{N_1, k_1\}$. Then for any $n \geq N$, we have

$$d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x) < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

where we have chosen a k such that $k > N$. □

Remark 2.3.7. Many students cannot write a careful proof of (3) of the Proposition. A beginner is advised to go through the proof once again. The trick of inserting n_k in the last inequality is an instance of what we call the 'curry leaves trick'. More on this can be found in Remark 6.1.2.

Definition 2.3.8. A metric space (X, d) is said to be *complete* if every Cauchy sequence in X converges to an element in X.

We say that a metric d is complete if the metric space (X, d) is complete.

Ex. 2.3.9. Show that the sequence (x_n) given by $x_n = 1/n$ is a Cauchy sequence in $(0, 1)$ (with the metric induced from that on $\mathbb{R}$) but is not convergent in $(0, 1)$. Hence conclude that $(0, 1)$ is not complete with respect to this metric.

Ex. 2.3.10. Show that any discrete metric space is complete.

Ex. 2.3.11. Let $D := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$. Is D complete?

Theorem 2.3.12 (Completeness of $\mathbb{R}$). *$\mathbb{R}$ is complete.*

Proof. This proof imitates that of Theorem 2.2.28.

Let (x_n) be a Cauchy sequence in $\mathbb{R}$. Let $\delta > 0$ be arbitrary. There exists a positive integer $N = N(\delta)$ such that for all $m \geq N$ and $n \geq N$, we have $|x_n - x_m| < \delta/2$. In particular we have $|x_n - x_N| < \delta/2$. Or, equivalently,

$$x_n \in (x_N - \delta/2, x_N + \delta/2) \qquad \text{for all } n \geq N.$$

From this we make the following observations:

(i) For all $n \geq N$, we have $x_n > x_N - \delta/2$.

(ii) If $x_n \geq x_N + \delta/2$, then $n \in \{1, 2, \ldots, N - 1\}$. Thus the set of n such that $x_n \geq x_N + \delta/2$ is finite.

We shall apply these two observations below for $\delta = 1$ and $\delta = \varepsilon$.

Let $S := \{x \in \mathbb{R} :$ there exists infinitely many n such that $x_n \geq x\}$. We claim that S is nonempty, bounded above and that $\sup S$ is the limit of the given sequence.

From (i), we see that $x_N - 1 \in S$. Hence S is nonempty.

From (ii) it follows that $x_N + 1$ is an upper bound for S. That is, we claim that $y \le x_N + 1$ for all $y \in S$. If this were not true, then there exists a $y \in S$ such that $y > x_N + 1$ and such that $x_n \ge y$ for infinitely many n. This implies that $x_n > x_N + 1$ for infinitely many n. This contradicts (ii). Hence we conclude that $x_N + 1$ is an upper bound for S.

By the LUB axiom, there exists $\ell \in \mathbb{R}$ which is $\sup S$. We claim that $\lim x_n = \ell$. Let $\varepsilon > 0$ be given. As ℓ is an upper bound for S and $x_N - \varepsilon/2 \in S$ (by (i)) we infer that $x_N - \varepsilon/2 \le \ell$. Since ℓ is the least upper bound for S and $x_N + \varepsilon/2$ is an upper bound for S (from (ii)) we see that $\ell \le x_N + \varepsilon/2$. Thus we have $x_N - \varepsilon/2 \le \ell \le x_N + \varepsilon/2$ or

$$|x_N - \ell| \le \varepsilon/2.$$

For $n \ge N$ we have

$$\begin{aligned} |x_n - \ell| &\le |x_n - x_N| + |x_N - \ell| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

We have thus shown that $\lim_{n\to\infty} x_n = \ell$. $\square$

Theorem 2.3.13. *$\mathbb{R}^n$ is complete.*

Proof. Observe that a sequence (x_k) is Cauchy in $\mathbb{R}^n$ iff the coordinate sequences $(x_{kj})_{k=1}^{\infty}$ is Cauchy in $\mathbb{R}$. Use the completeness of $\mathbb{R}$ to complete the proof. $\square$

Ex. 2.3.14. The notation is as in Ex. 1.2.73. Show that $(0, 1]$ is complete with respect to ρ while it is not complete with respect to the standard metric.

Ex. 2.3.15. Show that the metric space $\mathbb{C}$ with the metric $d(z, w) := |z - w|$ is complete.

Ex. 2.3.16. The notation is as in Ex. 1.2.72. Show that (X, d) is complete iff (X, δ) is complete.

Example 2.3.17. The notation is as in Ex. 2.1.19. We claim that $(\mathbb{Q}, d_p)$ is not complete. We shall prove this in the case when $p = 5$. (Recall that the symbol $a \equiv b(c)$ for integers means that the difference $a - b$ is divisible by c.) We construct a sequence (x_n) recursively with the following properties: (i) $x_n^2 + 1 \equiv (5^n)$ and (ii) $x_{n+1} \equiv x_n(5^n)$. Let $x_1 = 2$. Assume that we have already chosen x_k, $1 \le k \le n$. Let

$x_{n+1} = x_n + a5^n$ for some $a \in \mathbb{Z}$ to be determined. Our requirement is that $x_{n+1}^2 + 1 \equiv (5^{n+1})$, that is,

$$x_n^2 + 1 + 2x_n a 5^n + a^2 5^{2n} \equiv 0(5^{n+1}).$$

Since $x_n^2 + 1 \equiv 0(5^n)$, we need only find $a \in \mathbb{Z}$ such that $2x_n a + b \equiv 0(5)$, where $b = \frac{x_n^2+1}{5^n}$. Since 5 does not divide $2x_n$ (why?), such an $a \in \mathbb{Z}$ exists. The sequence (x_n) is Cauchy, since $x_n \equiv x_m(5^m)$ for any $m \leq n$. But it is not convergent. For, if $c \in \mathbb{Q}$ is the limit of the sequence (x_n), then $x_n^2+1 \to c^2+1$. On the other hand, by our construction, $x_n^2+1 \to 0$. Hence we conclude that $c^2 + 1 = 0$, which is impossible as $c \in \mathbb{Q}$.

Ex. 2.3.18. Let X be any metric space. Assume that (x_n) and (y_n) are Cauchy sequences in X. Show that $(d(x_n, y_n))$ is convergent in $\mathbb{R}$.

Remark 2.3.19. The concept of Cauchy sequences does not exist in an arbitrary topological space.

2.4 Bounded Sets

Definition 2.4.1. We define A to be *bounded* in (X, d) if there exists $x_0 \in X$ and $R > 0$ such that $A \subset B(x_0, R)$. See Figure 2.5.

Ex. 2.4.2. Show that a subset A of a metric space (X, d) is bounded iff for every $x \in X$ there exists $r > 0$ such that $A \subset B(x, r)$. (See Figure 2.6.)

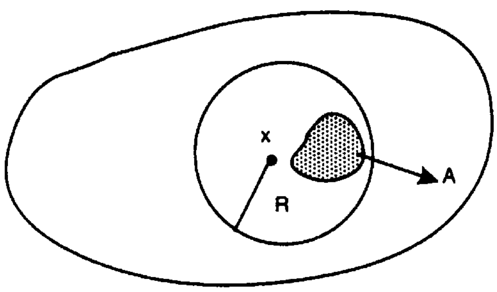

Figure 2.5: Bounded set

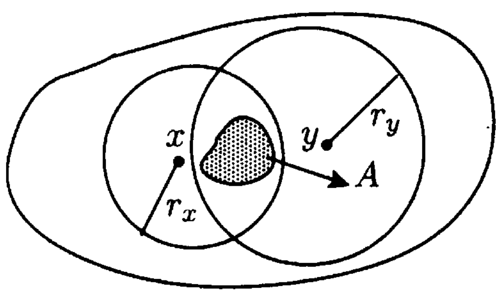

Figure 2.6: Illustration for Exercise 2.4.2

Ex. 2.4.3. In an NLS, A is bounded iff there exists $M > 0$ such that $\|v\| \leq M$ for all $v \in A$.

Ex. 2.4.4. Show that the union of a finite number of bounded sets in a metric space is bounded.

Definition 2.4.5. Let A be a nonempty subset of a metric space (X, d). The *diameter* diam (A) of A is defined by

$$\operatorname{diam}(A) := \sup\{d(x, y) : x, y \in A\},$$

in the extended real number system.

Ex. 2.4.6. Show that a subset A is bounded iff either it is empty or its diameter diam (A) is finite. (This is the standard definition of a bounded set in a metric space.)

Ex. 2.4.7. Show that the diameter $\operatorname{diam}(B(x, r)) \leq 2r$ and that the strict inequality can occur.

Ex. 2.4.8. Show that $\operatorname{diam}(B(x, r)) = 2r$ for any ball in an NLS. *Hint:* You may work with $\mathbb{R}^n$ and with the ball $B(0, r)$. Note that $x \in B(0, r)$ iff $-x \in B(0, r)$.

Ex. 2.4.9. Let (x_n) be a convergent sequence in a metric space (X, d). Show that the set $\{x_n\}$ is bounded. More generally, if (x_n) is Cauchy in X, show that the set $\{x_n\}$ is bounded in (X, d).

Ex. 2.4.10. Let $(X, \| \ \|)$ be an NLS. Show that $A \subset X$ is bounded iff there exists $M > 0$ such that $\|x\| \leq M$ for all $x \in A$.

Ex. 2.4.11. Which vector subspaces of an NLS are bounded subsets?

Ex. 2.4.12. Show that the set $O(n)$ of all orthogonal matrices (that is, the set of matrices satisfying $AA^t = I = A^tA$) is a bounded subset of $M(n, \mathbb{R})$. Here $M(n, R)$ is considered as an NLS as in Ex. 1.1.34.

Ex. 2.4.13. Show that the set $SL(n, \mathbb{R})$ of all $n \times n$ real matrices with determinant 1 is not bounded in $M(n, \mathbb{R})$. (The metric is as in Ex. 1.1.34.)

Ex. 2.4.14. Let G be a subgroup of the multiplicative group $\mathbb{C}^*$ of the non-zero complex numbers. Assume that as a subset of $\mathbb{C}$ it is bounded. Show that $|g| = 1$ for all $g \in G$.

Ex. 2.4.15. Consider $\mathbb{R}$ with the standard metric d and the metric $\delta(x, y) := \min\{d(x, y), 1\}$. Then $(\mathbb{R}, \delta)$ is bounded while $(\mathbb{R}, d)$ is not.

Remark 2.4.16. The moral of the last example is that 'boundedness' is metric specific. $\mathbb{R}$ with the standard metric is unbounded while with respect to $\delta := \min\{1, d\}$, it is bounded. However the topologies induced by d and δ are the same.

Remark 2.4.17. There is no concept of bounded sets in an arbitrary topological space.

2.5 Dense Sets

Definition 2.5.1. We say a subset $D \subset X$ of a metric space is *dense in* X if for any given $x \in X$ and $r > 0$, we have $B(x,r) \cap D \neq \emptyset$. In other words, any non-empty open set in X must contain a point of D.

A subset $D \subset X$ of a topological space is dense in X if for every nonempty open set $U \subset X$, we have $D \cap U \neq \emptyset$, that is U intersects D non-trivially.

A real life analogy will be useful. Imagine a lake with an abundant supply of fish. Then when you throw a net, however small, into the lake, the net will trap a fish. Then we say that the fish is dense in the lake.

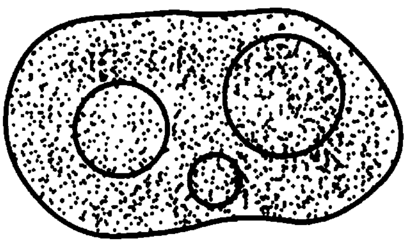

Figure 2.7: Dense set

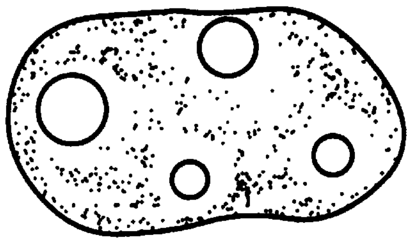

Figure 2.8: Not dense

Another example would be the set of people with tonsured head in Tirupati Balaji temple. In any 'small neighbourhood' of devotees in the temple, you will always find one with a tonsured head. (Do not stretch the real life examples too far!)

Ex. 2.5.2. Show that $\mathbb{Q}$ is dense in $\mathbb{R}$. Is $\mathbb{R} \setminus \mathbb{Q}$ dense in $\mathbb{R}$?
Can you think of a countable dense subset in $\mathbb{R}^2$? in $\mathbb{R}^n$?

Example 2.5.3. We now show that there exists a countable subset of the space ℓ_2 of Example 1.1.38 which is dense. If you have shown that $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$, a natural guess could be to think of sequences with rational terms only. But the set of all such sequences is $\mathbb{Q}^N$, that is, the set of all functions from $\mathbb{N}$ to $\mathbb{Q}$ which is uncountable. Why? There is one-one map of the set of functions from $\mathbb{N}$ to the two element set $\{0,1\}$ into the set of all functions from $\mathbb{N}$ to $\mathbb{Q}$. The former is in bijective correspondence with the set of all subsets of $\mathbb{N}$. Cantor's theorem says that there could be no onto map from a set X to its power set, that is, the set of its subsets. Hence it follows that $\mathbb{Q}^N$ is uncountable. It is quite feasible that $\mathbb{Q}^N \cap \ell_2$ is uncountable. For instance, consider the set

$$\prod_{n \in \mathbb{N}} \left([-2^{-n}, 2^{-n}] \cap \mathbb{Q}\right) \subset \ell_2.$$

So we modify our guess. We consider the set D_n of all sequences $x = (x_m)$ whose terms are rational and $x_k = 0$ for $k > n$. Let $D := \cup_{n \in \mathbb{N}} D_n$.

Each D_n is obviously bijective with $\mathbb{Q}^n$ and hence is countable. Thus D, being a countable union of countable sets, is itself countable.

We claim that D is dense in ℓ_2. We need to show that given any $x \in \ell_2$ and $\varepsilon > 0$, we can find $y \in D$ such that $\|x - y\|_2 < \varepsilon$. Since $x \in \ell_2$, there exists $N \in \mathbb{N}$ such that the tail $\sum_{N+1}^{\infty} |x_n|^2 < \varepsilon^2/2$. Consider $v := (x_1, \ldots, x_N) \in \mathbb{R}^N$. Since $\mathbb{Q}^N$ is dense in $\mathbb{R}^N$, there exists $r = (r_1, \ldots, r_N) \in \mathbb{Q}^N$ such that $\|v - r\|_2^2 < \varepsilon^2/2$. If we now define a sequence y such that $y_k = r_k$, $1 \leq k \leq N$ and $y_k = 0$ for $k > N$, then $y \in \ell_2$. Further we have

$$\|x - y\|_2^2 = \sum_{k=1}^{N} |x_k - r_k|^2 + \sum_{N+1}^{\infty} |x_k|^2 < \varepsilon^2/2 + \varepsilon^2/2 = \varepsilon^2.$$

Therefore, D is a countable dense subset of ℓ_2.

Ex. 2.5.4. Show that $D \subset X$ is dense in the metric space (X, d) iff every point of X is a limit point of D.

Ex. 2.5.5. Show that $D \subset X$ is dense in the (metric) space X iff its closure $\overline{D} = X$. (This is the standard definition.)

Lemma 2.5.6. *Let $S := \{n + m\sqrt{2} : n, m \in \mathbb{Z}\}$. Let $a, b \in \mathbb{R}$ be such that $a < b$. Then there exists an $s \in S$ such that $a < s < b$. In other words, S is dense in $\mathbb{R}$.*

Proof. If $x, y \in S$ and $k \in \mathbb{Z}$, then $x \pm y, kx \in S$. Let $n(m) := [m\sqrt{2}]$, the greatest integer less than or equal to $m\sqrt{2}$. Then, $0 \leq m\sqrt{2} - n(m) < 1$.

It is easy to see that if $n + m\sqrt{2} = n' + m'\sqrt{2}$, then $n = n'$ and $m = m'$.

Let $s_m := m\sqrt{2} - n(m)$. Then $0 \leq s_m < 1$ and $s_m \in S$. Also, if $m \neq m'$, then $s_m \neq s_{m'}$. Hence we conclude that $\{s_m : m \in \mathbb{Z}\}$ is an infinite subset of $S \cap [0, 1)$.

Given $\varepsilon > 0$, we partition $[0, 1)$ into k equal parts so that each subinterval has length less than ε. At least one of these subintervals must contain two distinct elements, say, $s_m, s_{m'}$ of $S \cap [0, 1)$. Without loss of generality let us assume that $s_m < s_{m'}$. Then we have $0 < s_{m'} - s_m < \varepsilon$. Since $s_{m'} - s_m \in S$, we have shown that given $\varepsilon > 0$, there exists an element $s \in S$ with $0 < s < \varepsilon$.

Now, let $\varepsilon > 0$ such that $b - a > \varepsilon$ be given. Then there exists $n \in \mathbb{Z}$ such that $a < n\varepsilon < b$. For, choose n to be the least integer k such that $k\varepsilon > a$. Then $(n-1)\varepsilon \leq a < n\varepsilon$. We claim that $n\varepsilon < b$. For, otherwise,

$$b - a \leq n\varepsilon - (n-1)\varepsilon = \varepsilon,$$

a contradiction.

We take $\varepsilon := (b-a)/2$. Then there exists $s \in S$ such that $0 < s < \varepsilon$. Hence there exists an integer n such that $a < ns < b$. Since $ns \in S$, the theorem is proved. $\square$

Ex. 2.5.7. Deduce from the foregoing lemma that the sum of two closed sets in $\mathbb{R}$ need not be closed.

Ex. 2.5.8. Does there exist a finite set which is dense in $\mathbb{R}$? What can you say about a metric space in which a finite set is dense?

Ex. 2.5.9. What are the dense subsets of a discrete (metric) space?

Ex. 2.5.10. Let (X, d) be a metric space. Assume that the only dense subset is X itself. Can you say something about the topology, that is, the family of open sets?

Ex. 2.5.11. Let A, B be two dense subsets of a (metric) space? Is $A \cup B$ dense? Is $A \cap B$ dense?

Ex. 2.5.12. If A, B are open dense subsets of a (metric) space X, is their intersection dense?

Ex. 2.5.13. Give an example of a proper open dense subset of $\mathbb{R}$.

Ex. 2.5.14. We know (from Lemma 1.2.40) that if U is an open subset of $\mathbb{R}$, then it is the union of a countable numbers of open intervals, say, $\{J_n\}$. (It is possible that $J_n = \emptyset$!) We define the "length" of U as the sum $\sum_{n=1}^{\infty} \ell(J_n)$. Given an example of an open dense set of finite length.

Ex. 2.5.15 (Weierstrass Approximation Theorem). The theorem states: Given any continuous function $f\colon [0,1] \to \mathbb{R}$ and given $\varepsilon > 0$, there exists a polynomial $p(x)$ such that $|f(x) - p(x)| < \varepsilon$ for all $x \in [0,1]$. Interpret this result using the concepts learnt so far.

We do not give a proof of this result. There are many proofs available and we refer the reader to [3] (page 143) or [4] (page 159) for elementary proofs.

2.6 Basis

Ex. 1.2.46 shows that in a metric space (X, d), a set U is open iff it is the union of a family of open balls. In fact, we can improve upon this. Consider the collection

$$\mathcal{B} := \{B(x, 1/n) : x \in X, n \in \mathbb{N}\}.$$

Then a set $U \subset X$ is open iff it is the union of members of $\mathcal{B}$. This collection has the following properties:

(i) X is the union of members of $\mathcal{B}$.

(ii) Given $B_1, B_2 \in \mathcal{B}$, their intersection $B_1 \cap B_2$ is the union of a class of members of $\mathcal{B}$.

All of what we want to do with the topology, that is, the class of open sets can be done with $\mathcal{B}$. For instance, x is a limit point of a set E iff for every $n \in \mathbb{N}$, $B(x, 1/n) \cap E \neq \emptyset$. Similarly, cluster points and other concepts that will be introduced later can be formulated using elements of $\mathcal{B}$. The advantage of this is that we need to check whatever we want to check only for a smaller class of open balls.

When $X = \mathbb{R}$, we can even take a smaller family which is countable:

$$\{(r - 1/n, r + 1/n) : r \in \mathbb{Q}, n \in \mathbb{N}\}.$$

A set in $\mathbb{R}$ is open iff it is the union of a family of members from this class.

These examples suggest that it may be expedient to make the following definition.

Definition 2.6.1. Let X be a (metric) space. We say that a family of open sets $\mathcal{B} := \{B_i : i \in I\}$ indexed by an indexing set I is a *basis* for the topology on X if it satisfies the condition:

Any open set in X is the union of some collection of members of $\mathcal{B}$.

Ex. 2.6.2. Let X be (metric) space and $D \subset X$. Let $\mathcal{B}$ be a basis for the topology on X. Then D is dense in X iff $D \cap B \neq \emptyset$ for every $B \in \mathcal{B}$.

In the sequel, we shall indicate whenever possible how to make use of this concept.

Ex. 2.6.3. Let X, Y be metric spaces. Consider the family of open sets of the form $\{B(x, \varepsilon) \times B(y, \varepsilon) : (x, y) \in X \times Y, \varepsilon > 0\}$. Show that this is a basis for the topology induced by the product metric on $X \times Y$.

2.7 Boundary of a Set

Definition 2.7.1. Let X be a (metric) space and $A \subset X$. A point $x \in X$ is said to be a *boundary point* of A in X if every open set that contains x intersects both A and $X \setminus A$ non-trivially. The *boundary* of A in X is the set of boundary points of A in X. We denote it by ∂A.

Example 2.7.2. Let us consider the following sets:

1. $A_1 = (a, b]$.
2. $A_2 = \mathbb{R} \setminus \{0\}$.
3. A_3 is the subset of $\mathbb{R}^2$ given by

$$\{x^2 + y^2 < 1, y > 0\} \cup \{-1 \leq x \leq 1, y = 0\} \cup \{x^2 + y^2 = 1, y < 0\}.$$

4. $A_4 = \{(x,y) : x^2 + y^2 = 1\}$.

The boundary points of $(a,b]$ are a and b. The only boundary point of A_2 is 0. The set of boundary points of A_3 is

$$\{(x,y) : x^2 + y^2 = 1\} \cup \{(x,y) : -1 \le x \le 1, y = 0\}.$$

The set of boundary points of A_4 is the set itself.

See Figures 2.9–2.12 and identify the boundary points.

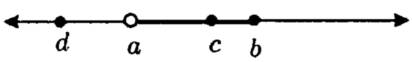

Figure 2.9: Boundary points of $(a,b]$

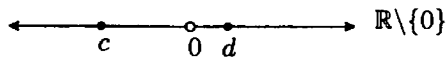

Figure 2.10: Boundary points of $\mathbb{R} \setminus \{0\}$

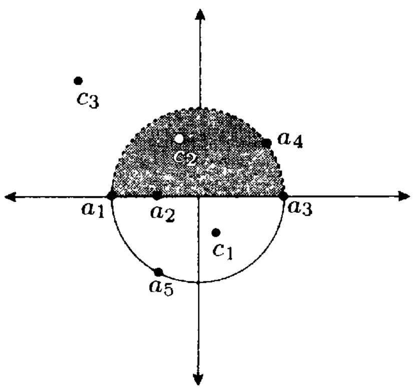

Figure 2.11: Boundary points of A_3

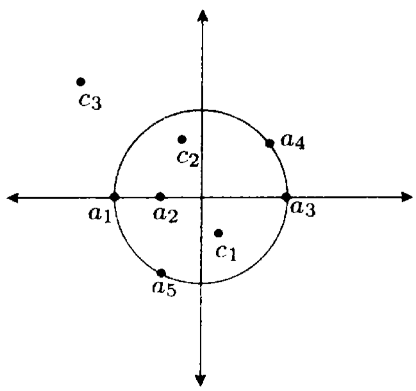

Figure 2.12: Boundary points of $x^2 + y^2 = 1$

Ex. 2.7.3. Let $X = \mathbb{R}$ and $A = (0,1]$. Show that $\partial A = \{0,1\}$.

Ex. 2.7.4. Consider $A = \mathbb{R} \times \{0\} \subset \mathbb{R}^2$. What is the boundary of A in $\mathbb{R}^2$?

Ex. 2.7.5. Show that the boundary of an open or closed ball in $\mathbb{R}^n$ is the sphere: $\partial B(x,r) = \partial B[x,r] = S(x,r) := \{y \in \mathbb{R}^n : d(x,y) = r\}$. Is this true in an NLS? in an arbitrary metric space?

Ex. 2.7.6. Let B be an open ball in $\mathbb{R}^n$. Find the boundary of B minus a finite number of points.

Ex. 2.7.7. Let $A := \{z \in \mathbb{C} : z = re^{it}, r \in [0,1], t \in (0, 2\pi)\}$. (Draw a picture.) Find the boundary of A.

Ex. 2.7.8. Let A be subset of a (metric) space. Show that $\partial A = \overline{A} \setminus A^0$. (This is the standard definition of the boundary of a set.)

Ex. 2.7.9. Let A be subset of a (metric) space. Is $\partial A = \partial(X \setminus A)$?

Chapter 3

Continuity

3.1 Continuous Functions

Definition 3.1.1. Let (X, d) and (Y, d) be metric spaces. A function $f\colon X \to Y$ is said to be *continuous at* $x \in X$ iff for every sequence (x_n) in X converging to x, we have $f(x_n) \to f(x)$.

We say that f is continuous on a subset $A \subset X$ if f is continuous at each $a \in A$.

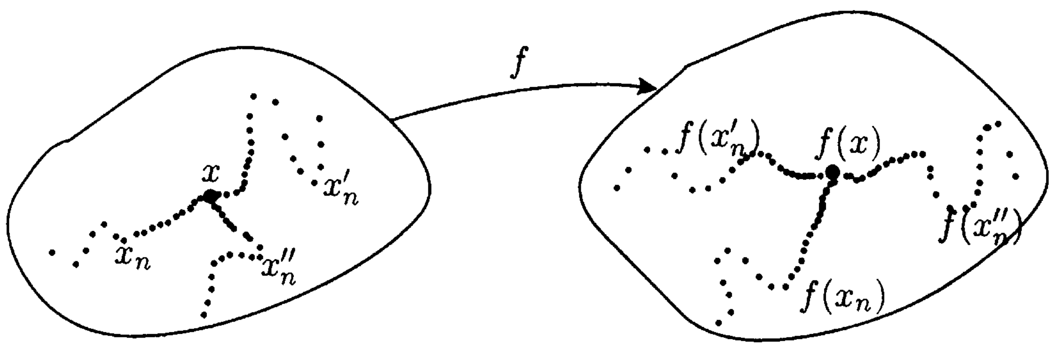

Figure 3.1: Continuity of f at x

Ex. 3.1.2. Show that any constant map from a metric space to another is continuous.

Ex. 3.1.3. Show that the identity map $x \mapsto x$ is continuous from a metric space (X, d) to itself.

Ex. 3.1.4. Show that the map $x \mapsto x^2$ from $\mathbb{R}$ to itself is continuous.

Example 3.1.5. We show that the maps $\mathbb{R}^2 \to \mathbb{R}$ given by $\alpha\colon (x, y) \mapsto x + y$ and $\mu\colon (x, y) \mapsto xy$ are continuous.

Let $(x, y) \in \mathbb{R}^2$ be arbitrary. We show that α is continuous at (x, y). Let $(x_n, y_n) \to (x, y)$. By Ex. 2.1.7, $x_n \to x$ and $y_n \to y$. By the standard result on algebra of limits of sequences in $\mathbb{R}$ (see also Ex. 2.1.9), it follows that $x_n + y_n \to x + y$. That is, $\alpha(x_n, y_n) \to \alpha(x, y)$. Hence α is continuous at (x, y).

A similar argument proves the continuity of μ on $\mathbb{R}^2$.

Ex. 3.1.6. Show that the projection maps $p_i\colon \mathbb{R}^n \to \mathbb{R}$ given by $p_i(x) = x_i$ where $x = (x_1, \ldots, x_n)$ is continuous.

Ex. 3.1.7. Let (X, d) and (Y, d) be metric spaces. Let us equip $X \times Y$ with the product metric. Let p_X and p_Y denote the projection of $X \times Y$ to X, respectively to Y given by $p_X(x, y) = x$, respectively $p_Y(x, y) = y$. Show that p_X and p_Y are continuous.

Ex. 3.1.8. Let the notation be as above. Fix $y_0 \in Y$. Let i_X denote the inclusion $x \mapsto (x, y_0)$. Show that i_X is continuous.

Theorem 3.1.9 (Space of Continuous Functions). *Let (X, d) be a metric space. Let $x \in X$.*

(i) *If $f, g\colon X \to \mathbb{R}$ are continuous at x, then so are $f + g, fg, af$, for any $a \in \mathbb{R}$. Consequently, the set of functions continuous at x form a real vector space.*

(ii) *Let $C(X, \mathbb{R})$ denote the set of all real valued continuous functions on X. Then they form a vector space over $\mathbb{R}$ under the obvious operations.*

(iii) *If $f\colon X \to \mathbb{R}$ is continuous at $x \in X$ and if $f(x) \neq 0$, then there exists $r > 0$ such that $f(x') \neq 0$ for all $x' \in B(x, r)$ and the function $g(x') := 1/f(x')$ from $B(x, r)$ to $\mathbb{R}$ is continuous at x.*

Analogous results hold if we replace $\mathbb{R}$ by $\mathbb{C}$.

Proof. Let $x_n \to x$. Then $(f + g)(x_n) = f(x_n) + g(x_n)$. We are given that $f(x_n) \to f(x)$ and $g(x_n) \to g(x)$. From the algebra of limits of sequences, it follows that $f(x_n) + g(x_n) \to f(x) + g(x) = (f + g)(x)$. This proves the continuity of $f + g$ at x.

The other statements are proved similarly. □

Remark 3.1.10. Do you appreciate the ease with which you proved these results thanks to our definition of continuity? The core of the argument falls upon established facts on convergent sequences in $\mathbb{R}$ or in $\mathbb{C}$!

Ex. 3.1.11. Use Ex. 3.1.6 and Theorem 3.1.9 to conclude that any polynomial function $p(x_1, \ldots, x_n)$ in the variables $x_1, \ldots, x_n$ will be continuous. (Give examples of such functions!)

Ex. 3.1.12. Consider $M(2,\mathbb{R})$. Let $f(A) = \det(A)$. Show that f is a continuous function. *Hint:* Ex. 3.1.11.

Can you think of a generalization?

Ex. 3.1.13. Fix $x \in X$. Show that the function f_x defined by $f_x(y) := d(x,y)$ is continuous.

Ex. 3.1.14. Find a continuous function $f\colon \mathbb{C}^* \to S^1 := \{z \in \mathbb{C} : |z| = 1\}$ such that $f(z) = z$ for $z \in S^1$.

Ex. 3.1.15. Show that the conjugation map $z \mapsto \overline{z}$ is continuous on $\mathbb{C}$.

Proposition 3.1.16. *Composite of continuous functions is continuous: Let X, Y, Z be metric spaces. Let $f\colon X \to Y$ be continuous at $x \in X$ and $g\colon Y \to Z$ be continuous at $y = f(x)$. Then the composite map $g \circ f\colon X \to Z$ is continuous at $x \in X$*

Proof. This is easy.

Let $x_n \to x$. Then $y_n := f(x_n) \to y = f(x)$ by the continuity of f at x. Since g is continuous at y, it follows that $g(y_n) \to g(y)$ or what is the same, $g(f(x_n)) \to g(f(x))$. $\square$

Ex. 3.1.17. In this exercise, the product sets are given the product metrics.

Show that the following maps are continuous:

(1) the vector addition map $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ given by$(x,y) \mapsto x + y$.
(2) the scalar multiplication map $\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ given by $(a,x) \mapsto ax$.
(3) the inner product map $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ given by $(x,y) \mapsto \langle x, y\rangle$.

Ex. 3.1.18. Let $f, g\colon X \to \mathbb{R}$ be continuous functions on a metric space X. Show that the map $\varphi\colon X \to \mathbb{R}^2$ given by $\varphi(x) = (f(x), g(x))$ is continuous.

Ex. 3.1.19. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be continuous. Show that the map $g\colon \mathbb{R}^2 \to \mathbb{R}$ given by $g(x,y) := f(x+y, x-y)$ is continuous.

Ex. 3.1.20. The map $A \mapsto A^t$ from $M(n,\mathbb{R})$ to $M(n,\mathbb{R})$ is continuous. Here A^t denotes the transpose of the matrix A such that the (i,j)-th entry of A^t is the (j,i)-th entry of A.

Ex. 3.1.21. Show that the squaring map $A \mapsto A^2$ on $M(n,\mathbb{R})$ is continuous. can you think of generalizations?

Ex. 3.1.22. Consider the set $X = \{1/n : n \in \mathbb{N}\} \cup \{0\} \subset \mathbb{R}$ with the induced metric. Show that the vector space of continuous functions on X is linearly isomorphic to the vector space of all convergent sequences in $\mathbb{R}$. *Hint:* Given $f \in C(X)$, consider the real sequence (x_n) where $x_n := f(1/n)$.

Ex. 3.1.23. Keep the notation as in the last exercise. Consider $Y = \mathbb{N} \cup \{\infty\}$. Let $f\colon Y \to X$ be the bijection defined by

$$f(n) = 1/n \text{ for } n \in \mathbb{N} \text{ and } f(\infty) = 0.$$

Using this bijection, we transfer the metric on X to Y. (See Ex. 1.2.73.) Let (A, d) be a metric space and (a_n) be a sequence in A. Let $a \in A$. Then $a_n \to a$ in A iff the function $\varphi\colon Y \to A$ defined by setting $\varphi(n) = a_n$ for $n \in \mathbb{N}$ and $\varphi(\infty) = a$ is continuous at ∞.

3.2 Equivalent Definitions of Continuity

In this section we shall define the standard ε-δ definition of continuity as well another one which involves open sets. The latter one generalizes to topological spaces. We shall also show how to use the latter one to give an 'easy' way of finding open or closed sets and allows us to decide whether a set is open or closed. The ε-δ definition leads us naturally to the definition of uniform continuity.

Theorem 3.2.1 (Equivalent Characterisations of Continuity). *Let X, Y be metric spaces. Let $f\colon X \to Y$ be a function. Then the following are equivalent:*

(a) *f is continuous at $x \in X$.*

(b) *Given any $\varepsilon > 0$, there exists a $\delta > 0$ such that if $d(x, x') < \delta$, then $d(f(x), f(x')) < \varepsilon$.*

(c) *Given an open set V containing $f(x)$ in Y, we can find an open set U containing x such that $f(U) \subset V$.*

Proof. We suggest that the reader draws pictures for each of the implications.

(a) $\implies$ (b): We shall prove this by contradiction. Assume that $\varepsilon > 0$ is given and that there exists no $\delta > 0$ with the required property. Thus, if we take $\delta = 1/n$, then there exists an x' such that $d(x, x') < 1/n$ but $d(f(x), f(x')) \geq \varepsilon$. Let us call this x' as x_n to emphasize its dependence on $\delta = 1/n$. As $d(x_n, x) < 1/n$, the sequence $x_n \to x$. (See Figure 3.2.) By our assumption, $f(x_n)$ must converge to $f(x)$. Hence we deduce that $d(f(x_n), f(x)) \to 0$. This does not happen, since $d(f(x_n), f(x)) \geq \varepsilon$ for all n. This is the desired contradiction.

(b) $\implies$ (c): Let V be given as in (c). Since V is open and $f(x) \in V$, there exists an $\varepsilon > 0$ such that $B(f(x), \varepsilon) \subset V$. Since we assume (b), for this ε, there exists $\delta > 0$ with the property stated in (b). That is, if we let $U := B(x, \delta)$, then U is an open set that contains x and is such that $f(U) \subset V$. (See Figure 3.3.)

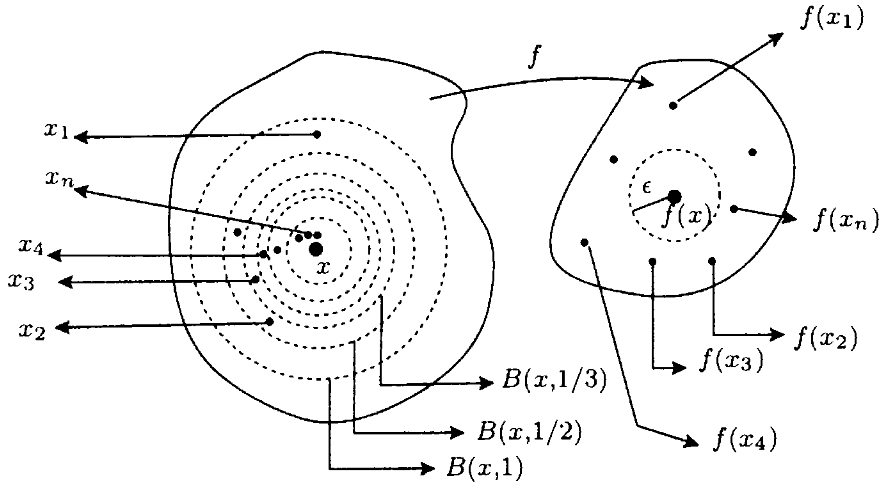

Figure 3.2: $a \implies b$ of Theorem 3.2.1

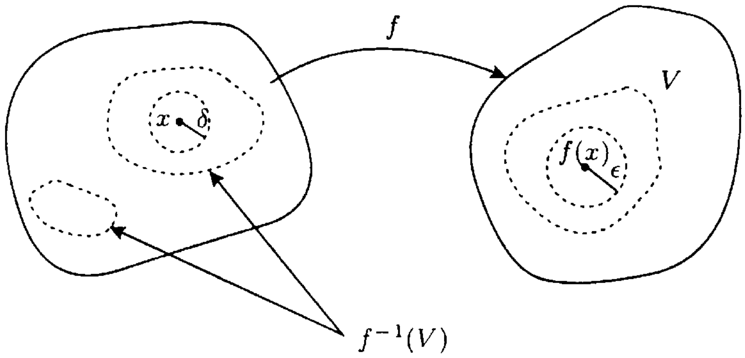

Figure 3.3: $b \implies c$ of Theorem 3.2.1

(c) $\implies$ (a): Let $x_n \to x$. We need to show that $f(x_n) \to f(x)$. Let $\varepsilon > 0$ be given. Consider $V := B(f(x), \varepsilon)$. This is an open set containing $f(x)$. So, by (c), there exists an open set U containing x such that $f(U) \subset V$. Since U is open and $x \in U$, there exists $r > 0$ such that $B(x,r) \subset U$. Since $x_n \to x$, there exists $N \in \mathbb{N}$ such that if $n \geq N$, we have $x_n \in B(x,r)$. It follows that $f(x_n) \in f(B(x,r)) \subset f(U) \subset B(f(x), \varepsilon)$ for all $n \geq N$. That is, $f(x_n) \to f(x)$. Therefore, f is continuous at x. $\square$

Ex. 3.2.2. To gain practice, show all possible two way implications of the last theorem: $(a) \iff (b)$, $(b) \iff (c)$, $(a) \iff (c)$.

Remark 3.2.3. Many students have problem with the ε-δ definition of continuity, that is, (b) of Theorem 3.2.1. A real life example may clarify this. Think of X as a set of ingredients or input and f as a process, procedure or a recipe which takes an input $x \in X$ and yields the output $f(x)$. We shall give more concrete examples presently. The end user or the consumer needs $y \in Y$. The manufacturer knows to get y, he needs the input x and by applying the process f to x he will get $y = f(x)$ and he can deliver the good to the consumer. But what happens in real life is that there is no guarantee the input will be exactly or precisely the x needed for the process f. The consumer also realizes this and he therefore sets a tolerance level of error. He says that what the manufacturer delivers is within this error tolerance, say, ε, (with respect to some measurement), he will accept the delivery. Now the onus is on the manufacturer. From his experience, he may know that if the input x' is within δ distance from x, the process will produce $f(x')$ which may meet the acceptance level of the consumer. Thus, the error tolerance ε of the output is given and then we find the error tolerance δ of the input.

Now a concrete situation. Let us imagine that a country A wants to bomb the secret laboratory of biochemical weapons of a country B. The scientists involved may say that if we launch a missile inclined at, say, 37° and at an initial velocity 1200 KM per hour, the missile will exactly land on the target. When they arrived at these numbers, they made a lot of simplifying assumptions. The missile is like a line rather than a three dimensional object. Nobody can have a perfect control over the initial velocity and the exact degree at which the missile is launched. What the president or the military of A wants is that knowing the destructive power of the bomb, the bomb should fall within 1 KM radius of the laboratory. Thus the ε is given. Now the scientists will find out that if the degree of the launch lies between 36° and 38° and if the initial launch speed is between 1180-1220, then they can meet the requirements. You can now think of various such situations. We hope that this example not only clarifies the answer to the questions "Which comes first, ε or δ in the definition of continuity" but also that this definition is exactly what is needed by the applied scientists and engineers!

Ex. 3.2.4. Let (X, d) be a metric space and $a \in X$. Assume that $f\colon X \to \mathbb{R}$ is a continuous function such that $f(a) > 0$. Then there exists $\delta > 0$ such that $f(x) > f(a)/2$ for all $x \in B(a, \delta/2)$. *Hint:* We needed this when we wish to prove that $\| \ \|_1$ satisfies $\|f\|_1 = 0$ iff $f = 0$ in Example 1.1.10. (See Figure 3.4.)

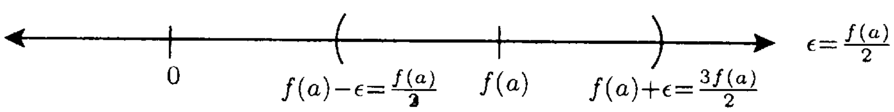

Figure 3.4: Illustration for Exercise 3.2.4

Part (c) of Theorem 3.2.1 suggests the following definition.

Definition 3.2.5. Let X, Y be topological spaces and $f\colon X \to Y$ be a map. We say that f is continuous at x if given an open set $V \ni f(x)$, we can find an open set $U \ni x$ such that $f(U) \subset V$.

Ex. 3.2.6. Show that Ex. 3.1.2, Ex. 3.1.3, Ex. 3.2.4, Theorem 3.1.9 and Proposition 3.1.16 remain valid (after suitable modifications in the statements!) for arbitrary topological spaces. (How will you modify Ex. 3.2.4?)

Ex. 3.2.7. Let X, Y be (metric) spaces. Show that a map $f\colon X \to Y$ is continuous iff for every open set $V \subset Y$, its inverse image $f^{-1}(V)$ is open in X.

Ex. 3.2.8. Let the notation be as in the last exercise. Let $\mathcal{B}_Y$ be basis of open sets for the topology on Y. (See Definition 2.6.1.) Show that f is continuous iff $f^{-1}(B)$ is open in X for all $B \in \mathcal{B}_Y$.

Ex. 3.2.9. Let X, Y be (metric) spaces. Show that a map $f\colon X \to Y$ is continuous iff for every closed set $V \subset Y$, its inverse image $f^{-1}(V)$ is closed in X.

Ex. 3.2.10. One can use Ex. 3.2.7 and Ex. 3.2.9 to show that certain sets are open or closed.

For instance the set of points $(x, y) \in \mathbb{R}^2$ such that $\cos(x^2) + x^3 - 47y > e^x - y^2$ is an open subset of $\mathbb{R}^2$. For, the function $\varphi(x, y) := \cos(x^2) + x^3 - 47y - (e^x - y^2)$ is continuous on $\mathbb{R}^2$ and the set under consideration is $f^{-1}(-\infty, 0)$.

As a concrete example, let us show that a rectangle $(a, b) \times (c, d)$ is open in $\mathbb{R}^2$. (This is was Ex. 1.2.34.) Where do we look for continuous functions? We look at the defining properties of the set. A point (x, y) lies in the rectangle iff $x \in (a, b)$ and $y \in (c, d)$. Now (a, b) and (c, d) are open and the projections $p_1(x, y) = x$ and $p_2(x, y) = y$ are continuous. Thus, the set $U_1 := p_1^{-1}(a, b)$ is open and $U_2 := p_2^{-1}(c, d)$ is open. The rectangle is the intersection of these open sets and hence is open.

Now redo Ex. 1.2.28, Ex. 1.2.29, Ex. 1.2.31, Ex. 1.2.34, Ex. 1.2.56 and Ex. 1.2.100.

Ex. 3.2.11. Show that the set of all invertible matrices in $M(2, \mathbb{R})$ is open. (Hint: Ex. 3.1.12.)

Ex. 3.2.12. Show that the set $SL(n, \mathbb{R})$ of matrices in $M(n, \mathbb{R})$ with determinant one is a closed subset of $M(n, \mathbb{R})$.

Ex. 3.2.13. Show that the set of all nipotent matrices in $M(n, \mathbb{R})$ is closed. (Recall a matrix $A \in M(n, \mathbb{R})$ is said to be nilpotent if $A^k = 0$ for some $k \in \mathbb{N}$.) *Hint:* Ex. 3.1.21. An infinite union of closed sets need not be closed. So, you need something from linear algebra also!

Ex. 3.2.14. Let (X, d) be a metric space. Consider the distance function $d: X \times X \to \mathbb{R}$. We equip $X \times X$ with the product metric (Ex. 1.1.32). We ask whether d is continuous.

Let (x_1, y_1) and (x_2, y_2) be given. We estimate

$$\begin{aligned}
|d(x_1, y_1) - d(x_2, y_2)| &\le |d(x_1, y_1) - d(x_2, y_1) + d(x_2, y_1) - d(x_2, y_2)| \\
&\le |d(x_1, y_1) - d(x_2, y_1)| \\
&\qquad + |d(x_2, y_1) - d(x_2, y_2)| \\
&\le d(x_1, x_2) + d(y_1, y_2) \\
&\le \delta((x_, y_1), (x_2, y_2)) + \delta((x_, y_1), (x_2, y_2)).
\end{aligned}$$

This shows that if $\varepsilon > 0$ is given, we may take $\delta \le \varepsilon/2$. Note that δ is independent of the point under consideration.

Ex. 3.2.15. Let X, Y be metric spaces. Let $f, g: X \to Y$ be continuous. Show that the set $E := \{x \in X : f(x) \neq g(x)\}$ is open in X. (Draw pictures.) *Hint:* Let $x \in E$. Choose open sets V and W in Y such that $f(x) \in V$ and $g(x) \in W$ and $V \cap W = \emptyset$. Then $x \in U := f^{-1}(V) \cap f^{-1}(W)$.

This exercise has an extension to arbitrary topological spaces provided that Y is Hausdorff.

Ex. 3.2.16. Let X, Y be metric spaces and $D \subset X$ be dense. Let $f, g: X \to Y$ be continuous functions such that $f(x) = g(x)$ for all $x \in D$. Show that $f = g$ on X.

This has an extension to topological spaces with an extra assumption on Y. Let X, Y be topological spaces. Assume that Y is Hausdorff. Let $D \subset X$ be dense in X. Let $f, g: X \to Y$ be continuous functions such that $f(x) = g(x)$ for all $x \in D$. Then show that $f = g$ on X. *Hint:* Last exercise.

Ex. 3.2.17. Let $f: \mathbb{R} \to \mathbb{R}$ be a continuous additive group homomorphism. Show that $f(x) = \lambda x$ for $x \in \mathbb{R}$ where $\lambda = f(1)$. *Hint:* Show by induction that $f(n) = nf(1)$ for any $n \in \mathbb{Z}$ and then that $f(m/n) = (m/n)f(1)$ for $m/n \in \mathbb{Q}$. Define $g(x) = f(1)x$. Show that $f = g$.

Ex. 3.2.18. Let D be dense in a (metric) space X. Let Y be another (metric) space. Assume that $f: X \to Y$ is continuous and onto. Show that $D' = f(D)$ is dense in Y.

Ex. 3.2.19. Let X, Y be metric spaces. Let $f, g: X \to Y$ be continuous. Then the set $\{x \in X : f(x) = g(x)\}$ is closed in X.

In particular, if $f: X \to X$ is continuous, the fixed point set $\{x \in X : f(x) = x\}$ is closed.

Remark: The results are false for general topological spaces. If we assume that Y is Hausdorff then the results are true. Compare Ex. 3.2.16.

Ex. 3.2.20. Define the map $f: \mathbb{R} \to \mathbb{R}$ by setting $f(x) = |x - 1|$. Show that f is continuous.

More generally, let $J = [-1, 1]$ and let

$$f(x) := \inf\{|x - t| : t \in [-1, 1]\}.$$

Show that f is continuous. Find an explicit expression/formula for f. Draw its graph. Can you interpret $f(x)$ as a geometric relation between x and the interval J?

Example 3.2.21. Let A be a nonempty subset of a metric space (X, d). Define

$$d_A(x) := \inf\{d(x, a) : a \in A\}, \qquad x \in X.$$

Then d_A is continuous. (Geometrically, we think of $d_A(x)$ as the distance of x to A.)

We give a proof even though it is easy, because of the importance of this result. Let $x, y \in X$ and $a \in A$ be arbitrary. We have, from the triangle inequality $d(a, x) \leq d(a, y) + d(y, x)$,

$$\begin{aligned} d(a, y) &\geq d(a, x) - d(y, x) \\ d(a, y) &\geq d_A(x) - d(y, x), \end{aligned} \tag{3.1}$$

since $d(a, x) \geq \inf\{d(a', x) : a' \in A\} := d_A(x)$. The inequality (3.1) says that $d_A(x) - d(y, x)$ is a lower bound for the set $\{d(a, y) : a \in A\}$. Hence the greatest lower bound of this set, namely, $\inf\{d(a, y) : a \in A\}$ is greater than or equal to this lower bound, that is,

$$d_A(y) \geq d_A(x) - d(y, x).$$

Therefore, $d_A(y) - d_A(x) \geq -d(y, x)$ or what is the same,

$$d_A(x) - d_A(y) \leq d(y, x).$$

If we interchange x and y in this inequality, we see that $\pm(d_A(x) - d_A(y)) \leq d(x, y)$, that is, $|d_A(x) - d_A(y)| \leq d(x, y)$. Continuity of d_A follows.

One may also arrive at this as follows:

$$d(a, x) \leq d(a, y) + d(y, x). \tag{3.2}$$

Now we are in the following situation. We have two families $A := \{a_i : i \in I\}$ and $B := \{b_i : i \in I\}$ of real numbers (bounded below) indexed by the same set I and with the property that $a_i \leq b_i$ for each $i \in I$. Let $a := \inf\{a_i\}$ and $b := \inf\{b_i\}$. Then we claim that $a \leq b$. For, $a \leq a_i$ for each i and hence $a \leq b_i$ for each i. Thus, a is a lower bound for the set B. Since b is the greatest lower bound for the set B, we conclude that $a \leq b$, as claimed. Applying this claim to (3.2), we obtain

$$d_A(x) \leq d_A(y) + d(y, x).$$

The rest of the argument is as above.

Remark 3.2.22. In view of Ex 3.1.13 and Example 3.2.21, we see that there exist non-constant real valued continuous functions on any metric space X. In fact, they are 'abundant' in the sense that given any two distinct points $x, y \in X$, there exists $f \colon X \to \mathbb{R}$ such that $f(x) = 0$ and $f(y) = 1$. Thus, any pair of distinct points are "separated" by means of a continuous function! In fact, we can say more. Any two disjoint closed subsets of a metric space can be separated by means of a continuous function. See Theorem 3.2.33.

In an arbitrary topological space, one cannot assert the existence of non-constant real valued continuous functions, leave alone being able to separate disjoint closed sets! We need to impose extra conditions to ensure such a possibility. This leads one to the definition of completely regular and normal spaces. As usual, we refrain from introducing these notions.

Ex. 3.2.23. Let $A = (0, 1) \subset \mathbb{R}$. Draw the graph of the function d_A.

Ex. 3.2.24. Let $x \in \mathbb{R}$. What is $d_{\mathbb{Q}}(x)$?

Ex. 3.2.25. Let $p := (a, b) \in \mathbb{R}^2$ and A be the x-axis. What is $d_A(p)$?

Ex. 3.2.26. What is $d_A(p)$ where $A := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$? Find an explicit expression.

Ex. 3.2.27. This exercise assumes knowledge of inner product spaces. Let V be a real (finite dimensional) inner product space and W a vector subspace. What is $d_W(x)$ for $x \in V$?

Ex. 3.2.28. Show that x is a limit point E iff $d_E(x) = 0$. Conclude that $x \in \overline{A}$ iff $d_A(x) = 0$. In particular, if A is closed, then $d_A(x) = 0$ iff $x \in A$.

Definition 3.2.29 (Distance between two subsets). Let A and B be two nonempty subsets of a metric space X. We define the distance $d(A, B)$ between them by setting $d(A, B) := \inf\{d(a, b) : a \in, b \in B\}$.

Ex. 3.2.30. Find two closed sets F_1 and F_2 which are disjoint but $d(F_1, F_2) = 0$. *Hint:* $\mathbb{N}$ and $\{n + \frac{1}{n} : n \in \mathbb{N}\}$.

Ex. 3.2.31. Find the distance $d(A, B)$ between A and B where

(a) $A = \mathbb{Q}$ and $B = \mathbb{R} \setminus \mathbb{Q}$.

(b) $A = \mathbb{Q}$ and B is any nonempty subset of $\mathbb{R}$.

(c) A is the rectangular hyperbola $xy = 1$ and B is the union of axes $xy = 0$?

Ex. 3.2.32. Let A be a subset in a metric space (X, d). Show that the set $\{x \in X; d_A(x) < \varepsilon\}$ is open for any $\varepsilon > 0$ (i) directly and (ii) by using the continuity of d_A.

Theorem 3.2.33 (Urysohn's Lemma). *Let A, B be two disjoint closed subsets of a metric space. There exists a continuous function $f: X \to \mathbb{R}$ such that $0 \le f \le 1$ and $f = 0$ on A and $f = 1$ on B.*

Proof. We want a continuous function that vanishes on A. So we consider d_A. We want it to be 1 on B, so we are tempted to consider d_A/d_A. This has problem. A little reflection immediately leads us to $f(x) := \frac{d_A(x)}{d_A(x)+d_B(x)}$. It is now an easy exercise for the student to show f is as we wanted.

First of note that $d_A(x) + d_B(x) \neq 0$ for any $x \in X$. For, if it were, each of the terms, being nonnegative, must be zero. But then this means that $x \in A$ and $x \in B$ by Ex. 3.2.28, a contradiction since A and B are disjoint. Hence we conclude that $f(x)$ makes sense for any $x \in X$. By algebra of continuous functions, f is also continuous. If $x \in A$, then $d_A(x) = 0$ so that $f(x) = 0$ for any $x \in A$. Also, if $x \in B$, then the denominator of $f(x)$ is $d_A(x) + d_B(x) = d_A(x)$ and hence $f(x) = 1$. Clearly, $0 \le f \le 1$. □

Ex. 3.2.34. Let A, B be nonempty subsets of a metric space (X, d).

(a) Show that the set $\{x \in X : d_A(x) < d_B(x)\}$ is open in X.

(b) Assume that A and B are closed disjoint subsets. Then there exist open sets $U \supset A$ and $V \supset B$ with $U \cap V = \emptyset$.

(c) Could you have deduced (b) from Urysohn's lemma also?

A topological space having the property mentioned in (b) is known as a normal space: any two disjoint closed sets can be 'separated' by means of open sets. But Urysohn's lemma says something stronger. They can be 'separated' by means of a real-valued continuous function.

Ex. 3.2.35. Let $A \in M(n, \mathbb{R})$. Consider vectors of $\mathbb{R}^n$ as column vectors, that is, as matrices of size $n \times 1$ so that the matrix multiplication Ax makes sense. Show that the map $x \mapsto Ax$ is continuous.

In fact, we can prove more. The joint map from $M(n, \mathbb{R}) \times \mathbb{R}^n$ to $\mathbb{R}^n$ given by $(A, x) \mapsto Ax$ is continuous.

Ex. 3.2.36. Let $(X, \| \ \|)$ be an NLS. Show that $\| \ \|: X \to \mathbb{R}$ is continuous.

Ex. 3.2.37. Let X, Y be NLS. Let $T: X \to Y$ be a linear map. Prove that T is continuous iff it is continuous at $0 \in X$. Use this to show that T is continuous iff there exists a constant $C > 0$ such that $\|Tx\| \leq C \|x\|$ for all $x \in X$.

Ex. 3.2.38. Let X be any NLS. Show that any linear map $T: \mathbb{R}^n \to X$ is continuous. *Hint:* Prove that $\|Tx\| \leq C \sum_{j=1}^{n} |x_j|$ for any $x \in \mathbb{R}^n$.

Ex. 3.2.39. Consider $\varphi(f) := f(0)$ as a map $\varphi: (C[0,1], \| \ \|_\infty) \to \mathbb{R}$. Show that φ is continuous.

Is the same map continuous if we equip $C[0,1]$ with the L^1-norm $\| \ \|_1$?

Ex. 3.2.40. Consider $C[0,1]$ with the norm $\| \ \|_\infty$ as in Ex. 1.1.27. Show that the map $f \mapsto \int_0^1 f(t)\, dt$ is continuous. Is the map still continuous if we take $\| \ \|_1$ as the norm on $C[0,1]$?

Ex. 3.2.41. Consider $C[0,1]$ with the norm $\| \ \|_\infty$ as in Ex. 1.1.27. Let Y be the vector subspace of all differentiable functions on $[0,1]$. Consider the linear map $D: (Y, \| \ \|_\infty) \to (C[0,1], \| \ \|_\infty)$ given by $Df = f'$, the derivative of f. Show that D is not continuous.

Ex. 3.2.42. Find a continuous function $f: (a,b) \to \mathbb{R}$ which is bijective and such that f^{-1} is also continuous.

Ex. 3.2.43. Let $f: \mathbb{R} \to \mathbb{R}$ be such that $f^{-1}(a, \infty)$ and $f^{-1}(-\infty, b)$ are open for any $a, b \in \mathbb{R}$. Show that f is continuous.

Ex. 3.2.44. Let A be a subset of a metric space (X,d). Let (Y,d) be another metric space. A function $f: X \to Y$ is said to be continuous on A iff f is continuous at each $a \in A$. Show that f is continuous on A iff $f: (A,d) \to (Y,d)$ is continuous.

Ex. 3.2.45. Let $f: (X,d) \to (Y,d)$ be continuous. Let $A \subset X$. Show that the restriction $f|_A$ of f to A is a continuous function from the metric space (A,d) to (Y,d). (Here the metric on A is the induced metric.)

Lemma 3.2.46 (Gluing Lemma). *Let X and Y be (metric) spaces.*

(1) *Let $\{U_i : i \in I\}$ be a family of open sets such that $\cup_i U_i = X$. Assume that there exists a continuous function $f_i: U_i \to Y$ for each $i \in I$ with the property that $f_i(x) = f_j(x)$ for all $x \in U_i \cap U_j$ and $i, j \in I$. Then the function $f: X \to Y$ defined by setting $f(x) := f_i(x)$ if $x \in U_i$ is well-defined and continuous on X.*

(2) *Let* $\{A_i : i \in F\}$ *be a finite family of closed sets such that* $\cup_i A_i = X$. *Assume that there exists a continuous function* $f_i\colon A_i \to Y$ *for each* $i \in I$ *with the property that* $f_i(x) = f_j(x)$ *for all* $x \in A_i \cap A_j$ *and* $i, j \in F$. *Then the function* $f\colon X \to Y$ *defined by setting* $f(x) := f_i(x)$ *if* $x \in A_i$ *is well-defined and continuous on* X.

Proof. The proofs are straight forward and the reader should attempt them on his own. The only hint needed is that if $U \subset X$ is open (respectively closed) and $A \subset U$ is open (respectively closed) *in* U, then A is open (respectively closed) in X.

Let us prove (1). Let $V \subset Y$ be open. Then we have to show that $f^{-1}(V)$ is open in X. Now, we observe $f^{-1}(V) \cap U_i = f_i^{-1}(V) \cap U_i$. For, $x \in f^{-1}(V) \cap U_i$ iff $x \in U_i$ and $f(x) \in V$, that is, iff $x \in U_i$ and $f_i(x) \in V$, that is iff $x \in U_i \cap f_i^{-1}(V)$. Since $f_i\colon U_i \to Y$ is continuous, the set $f_i^{-1}(V) \cap U_i$ is open in U_i and hence open in X. Thus, it follows that $f^{-1}(V)$ is the union of open sets and hence is open:

$$f^{-1}(V) = f^{-1}(V) \cap X = f^{-1}(V) \cap (\cup_i U_i) = \cup_i \left(f_i^{-1}(V) \cap U_i\right).$$

This proves the continuity of f on X.

The proof of (2) is very similar, except that we use the characterization of continuity by means of inverse images of closed sets. Let $C \subset Y$ be a closed set. We shall show that $f^{-1}(C)$ is closed in X. As in the earlier case, we find that $f^{-1}(C) \cap A_i = f_i^{-1}(C) \cap A_i$ and that it is closed in A_i and hence in X. We then express $f^{-1}(V)$ as a finite union of closed sets of this form.

$$f^{-1}(C) = f^{-1}(C) \cap X = f^{-1}(C) \cap (\cup_i A_i) = \cup_i \left(f_i^{-1}(C) \cap A_i\right).$$

This completes the proof. □

An immediate, though trivial, application is

Ex. 3.2.47. Show that the absolute-value function $|\ |: \mathbb{R} \to \mathbb{R}$ is continuous.

A most useful application is the following

Ex. 3.2.48. Let $f, g\colon [0,1] \to X$ be continuous. Assume that $f(1) = g(0)$. Define

$$h(t) := \begin{cases} f(2t) & 0 \le t \le 1/2 \\ g(2t-1) & 1/2 \le t \le 1. \end{cases}$$

Show that h is continuous. (To see this exercise in proper perspective, see Ex. 5.2.4.)

A beautiful application of the gluing lemma is the Tietze extension theorem.

Theorem 3.2.49 (Tietze). *Let A be a closed subset of a metric space X. Given a continuous function $g\colon A \to [0,1]$ there exists a continuous function $f\colon X \to [0,1]$ which is an extension of g, that is, $f(a) = g(a)$ for all $a \in A$.*

Proof. By considering the function $x \mapsto 1 + g(x)$, we may assume that $g\colon X \to [1,2]$.

We define

$$f(x) := \begin{cases} g(x) & \text{if } x \in A \\ \frac{\inf\{g(a)d(a,x) : a \in A\}}{d_A(x)} & \text{if } x \notin A. \end{cases}$$

Note that since $0 \le g(a) \le 2$ for $a \in A$, we have $1 \le f(x) \le 2$ for $x \in X \setminus A$. Hence $f\colon X \to [1,2]$ is an extension of g.

We plan to show that f is continuous on the closed sets A and $\overline{X \setminus A}$. Since $X = A \cup \overline{X \setminus A}$, the result will follow from the gluing lemma. Since $f = g$ on A, the continuity on A is clear. We need only establish the continuity of f on $\overline{X \setminus A}$.

Case 1. Let $x \in X \setminus A$. Since d_A is continuous at x and since $d_A(x) > 0$ (why?), to prove the continuity of f at x, it suffices to show that $h\colon u \mapsto \inf\{g(a)d(a,u) : a \in A\}$ is continuous at x. (Why?)

Let $\varepsilon > 0$ be given. Let $u \in X \setminus A$ with $d(u,x) < \varepsilon/2$. (Draw pictures.) Then $a \in A$ implies

$$d(x,a) \le d(x,u) + d(u,a) < \frac{\varepsilon}{2} + d(u,a).$$

Multiplying the inequality by $g(a)$ and using the fact that $1 \le g(a) \le 2$, we get

$$g(a)d(x,a) < \varepsilon + g(a)d(u,a).$$

Taking the infimum as $a \in A$ yields

$$h(x) \le \varepsilon + h(u).$$

Similarly, we show that $h(u) \le \varepsilon + h(x)$. Thus, $|h(u) - h(x)| \le \varepsilon$ for $u \in (X \setminus A) \cap B(x,\varepsilon)$. This proves that f is continuous at $x \in X \setminus A$.

Case 2. Let x be such that every open ball centered at x intersects both A and $X \setminus A$, that is, $x \in \partial A$. Let $\varepsilon > 0$ be given. By the continuity of g at x, there exists a $\delta > 0$ such that

$$|g(a) - g(x)| < \varepsilon \text{ for all } a \in A \cap B(x,\delta). \tag{3.3}$$

In particular,

$$|f(x) - f(u)| < \varepsilon \text{ for all } u \in A \cap B(x, \delta/4). \tag{3.4}$$

We intend to prove the same thing holds for $u \in (X \setminus A)$. See Figure 3.5.

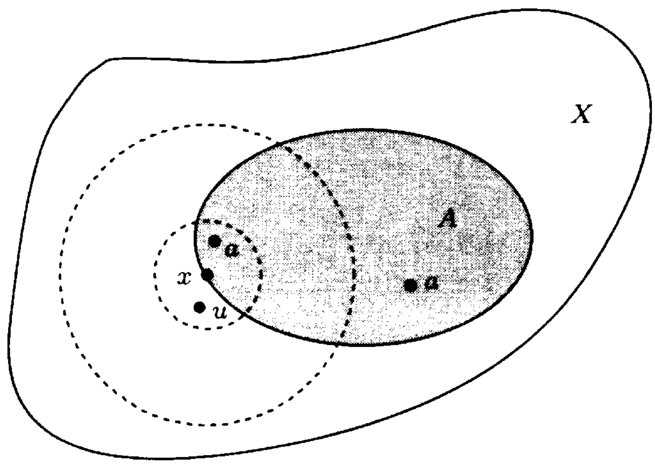

Figure 3.5: Tietze extension theorem

Let $u \in (X \setminus A) \cap B(x, \delta/4)$. We claim

$$h(u) := \inf\{g(a)d(a,u) : a \in A\} = \inf\{g(a)d(a,u) : a \in A \cap B(x,\delta)\}. \tag{3.5}$$

In fact, $a \notin A \cap B(x, \delta)$ implies

$$d(u,a) \geq d(x,a) - d(x,u) > \delta - \frac{\delta}{4} = \frac{3\delta}{4},$$

so that

$$\inf\{g(a)d(u,a) : a \in A \cap B(x,\delta)\} \geq \frac{3\delta}{4} \text{ as } g \geq 1. \tag{3.6}$$

On the other hand, since $x \in A \cap B(x, \delta)$, we deduce

$$g(x)d(x,u) \leq 2d(x,u) < 2 \cdot \frac{\delta}{4} = \frac{\delta}{2} < \frac{3\delta}{4}. \tag{3.7}$$

The claim (3.5) follows from (3.6)-(3.7). Note also that a similar but easier argument shows that

$$\inf\{d(u,a) : a \in A \cap B(x,\delta)\} = \inf\{d(u,a) : a \in A\} \equiv d_A(u). \tag{3.8}$$

From (3.3), we have

$$g(x) - \varepsilon < g(a) < g(x) + \varepsilon \text{ for } x \in A \cap B(x, \delta).$$

Hence,

$$(g(x)-\varepsilon)\,d_A(u) \le \inf\{g(a)d(u,a) : a \in A \cap B(x,\delta)\} \le (g(x)+\varepsilon)\,d_A(u).$$

Here we have used (3.8) to obtain the middle inequality. Using (3.5), we conclude that $|f(u)-g(x)| \le \varepsilon$. Since $x \in A$, we have $g(x) = f(x)$ so that we have shown

$$|f(u)-f(x)| < \varepsilon \text{ for all } u \in (X \setminus A) \cap B(x, \delta/4).$$

Thus f is continuous at $x \in \partial A$. This completes the proof of the theorem. □

Ex. 3.2.50. Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be defined as follows:

$$f(x) = \begin{cases} x & \text{if } \|x\| \le 1 \\ \frac{x}{\|x\|^2} & \text{if } \|x\| > 1. \end{cases}$$

Show that f is continuous.

It is well-known that it is impossible to make a continuous choice $\theta(z) \in \arg(z)$ on $\mathbb{C}^*$. That is, there is no continuous map $\theta\colon \mathbb{C}^* \to \mathbb{R}$ such that $z = |z|\exp(\theta(z))$ for $z \in \mathbb{C}^*$. We shall see a proof of this statement later.

However, the following lemma says that it is possible to assign the argument of a complex number in a continuous fashion if we restrict ourselves to $\mathbb{C}$ minus $\{z \in \mathbb{C} : \operatorname{Re} z \le 0\}$, or the complex plane minus any closed half line starting from the origin.

Lemma 3.2.51. *There exists a continuous map*

$$\alpha\colon X := \mathbb{C} \setminus \{z \in \mathbb{C} : z \in \mathbb{R} \textit{ and } z \le 0\} \to (-\pi, \pi)$$

such that $z = |z|\, e^{i\alpha(z)}$ for all $z \in X$.

Proof. We shall give a sketch of a proof.

Let us define the following open half-planes whose union is X: $H_1 := \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$, $H_2 := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ and $H_3 := \{z \in \mathbb{C} : \operatorname{Im} z < 0\}$. We define α_i on H_i which glue together to give the required map.

Let $z \in H_1$. Then $\operatorname{Re} z = |z|\cos\theta$ for some $\theta \in [-\pi, \pi]$ and hence $\cos\theta > 0$. Hence $\theta \in (-\pi/2, \pi/2)$. The sine function is increasing on $(-\pi/2, \pi/2)$ so that we have the continuous inverse $\sin^{-1}\colon (-1,1) \to (-\pi/2, \pi/2)$. We define $\alpha_1(z) := \sin^{-1}(\frac{\operatorname{Im} z}{|z|})$. We can similarly define $\alpha_2\colon H_2 \to (0, \pi)$ and $\alpha_3\colon H_3 \to (-\pi, 0)$ by

$$\alpha_2(z) = \cos^{-1}\left(\frac{\operatorname{Re} z}{|z|}\right) \quad \text{and} \quad \alpha_3(z) = \cos^{-1}\left(\frac{\operatorname{Re} z}{|z|}\right).$$

One easily sees that they agree upon their common domains. Thus we get the required function α. □

3.3 Topological Property

Definition 3.3.1. Let X, Y be (metric) spaces. A map $f\colon X \to Y$ is said to be a *homeomorphism* if f is a bijection and both $f\colon X \to Y$ and $f^{-1}\colon Y \to X$ are continuous. We say that two topological spaces X and Y are homeomorphic if there exists a homeomorphism from X onto Y.

Homeomorphism is something like an 'isomorphism' in the theory of groups or 'linear isomorphism' between vector spaces.

A property P of a space is said to be a *topological property* if homeomorphic spaces share the same properties.

For example, the metric spaces $(0, 1]$ and $[1, \infty)$ (with the standard induced metric) are homeomorphic via the map $x \mapsto 1/x$. The first one is bounded while the second one is not. Therefore, we conclude that the property that a metric space is bounded in not a topological property.

Similarly, any Cauchy sequence in $[1, \infty)$ converges to a point in $[1, \infty)$ while there exists a Cauchy sequence in $(0, 1]$ which is not convergent in $(0, 1]$. Thus the property that every Cauchy sequence in a metric space is convergent is not a topological property.

Are there any properties that are topological? Existence of a countable basis of open sets is a topological property. Later, we shall see that 'compactness', 'connectedness' and 'path-connectedness' are topological properties. See also Ex. 3.3.15.

Ex. 3.3.2. Show that two metrics d_1, d_2 on a set X are equivalent iff the identity map $I_X\colon (X, d_1) \to (X, d_2)$ is a homeomorphism.

Ex. 3.3.3. Show that homeomorphism is an equivalence relation: if $X$is homeomorphic to Y and Y is homeomorphic to Z, then X is homeomorphic to Z.

Ex. 3.3.4. Show that any two closed and bounded intervals in $\mathbb{R}$ are homeomorphic.

Example 3.3.5. Let S^n be the unit sphere in $\mathbb{R}^{n+1}$ defined by

$$S^n := \left\{x := (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} x_i^2 = 1\right\}.$$

Let $p := e_{n+1}$ be the north pole $(0, 0, \dots, 0, 1)$ and $q := (0, \dots, 0, -1)$ be the south pole. Let $U := S^n \setminus \{p\}$ and $V := S^n \setminus \{q\}$. We show that $S^n \setminus \{p\}$ is homeomorphic to $\mathbb{R}^n$.

The *stereographic projection* φ from the north pole is the map that sends a point $x \in S^n \setminus \{p\}$ to the point of intersection of the line joining p and x with the equator $x_{n+1} = 0$. Similarly the stereographic projection ψ from the south pole is defined. See Figure 3.6 on 73.

We shall now find a coordinate expression for

$$\varphi^{-1} : \mathbb{R}^n = \{x \in \mathbb{R}^{n+1} : x_{n+1} = 0\} \to S^n \setminus \{p\}.$$

Let $x = (x_1, \dots, x_n, 0) \in \mathbb{R}^n$. Then the line joining x with $e_{n+1} = p$ is given by

$$\alpha(t) = tx + (1-t)p = (tx', 1-t)$$

where $x' = (x_1, \dots, x_n)$. Now $\alpha(t)$ lies on S^n if and only if $\|\alpha(t)\|^2 = 1$ if and only if $t^2 \|x'\|^2 + (1-t)^2 = 1$ if and only if $t = 2(1 + (\|x'\|)^2)^{-1}$. In this case

$$\alpha(t) = (1 + (\|x'\|^2)^{-1}(2x_1, \dots, 2x_n, \|x'\|^2 - 1).$$

Thus we see that

$$\varphi^{-1}(x) = \alpha(t) = \frac{1}{(1 + \|x'\|^2)}(2x_1, \dots, 2x_n, \|x'\|^2 - 1). \tag{3.9}$$

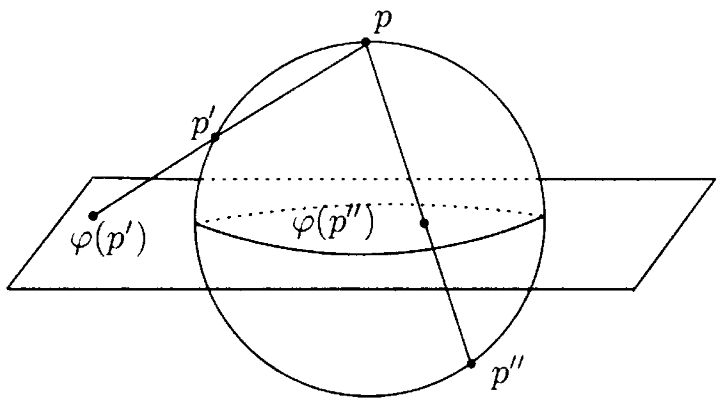

Figure 3.6: Stereographic projection

In a similar way one can show that

$$\psi^{-1}(x) = \alpha(t) = \frac{1}{(1 + \|x'\|^2)}(2x_1, \dots, 2x_n, 1 - \|x'\|^2).$$

It is obvious from the expressions for φ^{-1} and ψ^{-1} that they are homeomorphisms. ☐

Example 3.3.6. We show that the closed ball $B := B[0,1] \subset \mathbb{R}^n$ is homeomorphic to the cube $Q := [-1,1]^n \subset \mathbb{R}^n$.

Note that $Q := B_{d_\infty}[0,1]$, the closed unit ball in the metric induced by the max norm $\| \ \|_\infty$. We also observe that $B \subset Q$. For, if $\|x\| \le 1$, then $|x_i| \le 1$ for $1 \le i \le n$ and hence $\|x\|_\infty := \max\{|x_i| : 1 \le i \le n\} \le 1$. Let $K := \{x \in \mathbb{R}^n : \|x\|_\infty = 1\}$ be the boundary of Q. Let

$x \in Q$ be nonzero. Then the ray $\{tx : t \geq 0\}$ meets the boundary K exactly at one point. (Why?) Call this point of intersection $g(x)$. Can we find this point explicitly? (It is precisely the unit vector (in the norm $\| \ \|_\infty$) in the direction of x.) We have $\|tx\|_\infty = t\|x\|_\infty = 1$ iff $t := \frac{1}{\|x\|_\infty}$. Hence $g(x) := \frac{x}{\|x\|_\infty}$. Since $\|x\|_\infty \leq \|x\|$, the map $x \mapsto \|x\|_\infty$ is continuous on $(\mathbb{R}^n, \| \ \|)$. Therefore, the induced map $Q \setminus \{0\} \to K$ given by $x \mapsto g(x) := \frac{x}{\|x\|_\infty}$ is continuous.

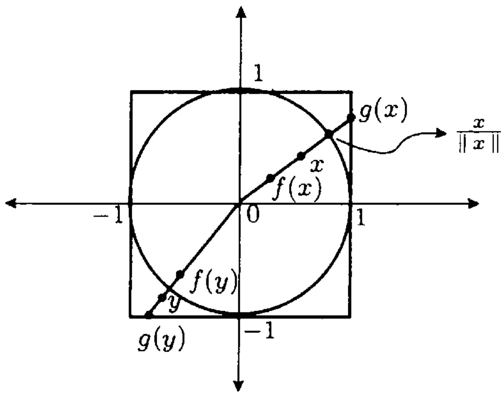

Figure 3.7: Homeomorphism of $[-1,1]^n$ and $B[0,1]$

Now the strategy of the construction of a homeomorphism is this. We map the line segment $[0, g(x)]$ onto the line segment $[0, \frac{x}{\|x\|}]$. We let $u(x) := \frac{x}{\|x\|}$ be the Euclidean unit vector in the direction of x. Thus we want to map $[0, g(x)]$ bijectively on $[0, u(x)]$. We know how to achieve this! Thus we have arrived at the map $f\colon Q \to B$ given by

$$f(x) := \begin{cases} \frac{x}{\|g(x)\|} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

The continuity of f at $x \neq 0$ follows from that of g. At $x = 0$, we have

$$\|f(x) - f(0)\| = \|f(x)\| = \left\| \frac{x}{\|g(x)\|} \right\| = \frac{\|x\|}{\|g(x)\|} \leq \|x\|,$$

since $\|g(x)\| \geq 1$. Clearly, f is a bijection. (Why?) The inverse of f is given by

$$f^{-1}(v) = \begin{cases} \|g(v)\| v & \text{if } v \in B \text{ and } v \neq 0 \\ 0 & \text{if } v = 0. \end{cases}$$

Again, the continuity of f^{-1} is clear. Thus f is a homeomorphism. □

Ex. 3.3.7. Show that $(0,1]$ (with the subspace topology induced from $\mathbb{R}$) is homeomorphic to $[1,\infty) \subset \mathbb{R}$.

Ex. 3.3.8. Any two open balls in $\mathbb{R}^n$ are homeomorphic.

Ex. 3.3.9. Show that a circle and an ellipse in $\mathbb{R}^2$ are homeomorphic.

Ex. 3.3.10. Show that $\mathbb{R}$ and the parabola given by $y = x^2$ are homeomorphic.

Remark 3.3.11. We shall see later that no two of a circle $x^2 + y^2 = 1$, a parabola $y = x^2$ and a hyperbola $x^2 - y^2 = 1$ are homeomorphic. (See Ex. 4.1.31 on page 90, Ex. 5.1.39 and Ex. 5.1.40 on page 114.)

Ex. 3.3.12. Let $\mathbb{N} \subset \mathbb{R}$ be given the induced metric d. Consider $\mathbb{N}$ with discrete metric δ. Are $(\mathbb{N}, d)$ and $(\mathbb{N}, \delta)$ homeomorphic?

Ex. 3.3.13. Two metrics d_1, d_2 on a set X are equivalent iff the identity map of X from (X, d_1) to (X, d_2) is a homeomorphism.

Ex. 3.3.14. Show that $(-1, 1)$ is homeomorphic to the parabola $y = x^2$ in $\mathbb{R}^2$. *Hint:* Ex. 3.3.3.

Ex. 3.3.15. A (metric) space having a countable dense subset is a topological property.

Ex. 3.3.16. Use Ex. 3.3.13 and Ex. 3.2.37 to solve Ex. 1.2.69.

3.4 Uniform Continuity

Consider the function $f\colon (0,1) \to \mathbb{R}$ given by $f(x) = 1/x$. It is easy to show that f is continuous at each $a \in (0,1)$. What we would like to analyze is the behaviour of δ required to prove the continuity of f at x as x varies in $(0,1)$. As $x \to 0$, we see that $f(x)$ goes to 'infinity'. Thus, it is intuitively clear that if we want to control the value of $f(x')$ for x' near to x, nearer x to 0, smaller the value of δ. That is, if we want to assert that $f(x')$ is within ε-distance of $f(x)$, then we may have to restrict x' to smaller and smaller open intervals around x as x goes nearer and nearer to 0. It follows therefore that given $\varepsilon > 0$, we cannot make a single choice of δ which will work for all $x \in (0,1)$. This discussion motivates the following definition.

Definition 3.4.1. Let $f\colon (X, d) \to (Y, d)$ be a function. We say that f is *uniformly continuous* on X if for a given $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever $x_1, x_2 \in X$ are such that $d(x_1, x_2) < \delta$, we have $d(f(x_1), f(x_2)) < \varepsilon$.

Note that while the continuity of f at a point is discussed, we are concerned only with the values (behaviour) of the function near the point under consideration. But when we wish to say that a function is uniformly continuous on its domain, we need to know the values of f on the entire domain. Hence the continuity is known as a local concept whereas the uniform continuity is known as a global concept.

The best way to understand uniform continuity is to look at some examples and see how the behaviour of the function on the *entire domain* plays a role.

Example 3.4.2. Let us look at a most standard example: $f\colon [1,\infty) \to \mathbb{R}$ given by $f(x) = 1/x$. We estimate

$$\begin{aligned} |f(x) - f(y)| &= \frac{|y-x|}{xy} \\ &\leq |x-y|, \text{ since } x \geq 1 \text{ and } y \geq 1. \end{aligned}$$

So, if $\varepsilon > 0$ is given, then we may choose $\delta = \varepsilon$.

On the other hand, look at $g\colon (0,1) \to \mathbb{R}$ given by $g(x) = 1/x$. We expect trouble at points near to 0 and hence guess that g is not uniformly continuous on $(0,1)$. The natural choice of points near to zero are $1/n$. Let us see what happens when we consider $x = 1/n$ and $y = 1/m$, $n \neq m$. We have

$$|f(x) - f(y)| = |n - m| \geq 1.$$

This shows that g cannot be uniformly continuous on $(0,1)$. For, if it were, for $\varepsilon = 1$, there exists $\delta > 0$ such that

$$|x-y| < \delta \implies |f(x) - f(y)| < 1.$$

Given such a δ, we choose $n > 1/\delta$ and let $x = 1/n$ and $y = \frac{1}{2n}$. Then $|x-y| < \delta$ but $|f(x) - f(y)| = n$.

Example 3.4.3. We now look at $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$. If we draw the graph of the function, we may observe that if $|x|$ is very large, small increments in x produce large differences in the values of f taken at these points. So we guess that the function is not uniformly continuous on $\mathbb{R}$. To prove this rigorously, we have to choose points which are 'near to infinity'. The obvious choices are $\pm n \in \mathbb{N}$. Let us see what happens.

$$|f(n+h) - f(n)| = 2nh + h^2 \geq 2nh.$$

To expect that for a given $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $|h| < \delta$, then $|f(x+h) - f(x)| < \varepsilon$ is unrealistic, since in that case we are asserting that $|nh| < \varepsilon$ for all n! This is blatant violation of the

Archimedean property of $\mathbb{R}$. This thinking aloud suggests the following proof. Let $\varepsilon = 1$ and assume if possible that there exists $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < 1$. Consider $x = n$ and $y = n + \frac{1}{n}$. If n is sufficiently large, then $|x - y| = 1/n < \delta$. But we have

$$|f(x) - f(y)| = 2 + (1/n)^2 \geq 2!$$

Our analysis also suggests that f may be uniformly continuous on any finite interval. We leave its proof as an exercise.

Ex. 3.4.4. Prove that the following functions are uniformly continuous:
(a) $f\colon [1, \infty) \to \mathbb{R}$ given by $f(x) = 1/x^n$, for $n \in \mathbb{N}$.
(b) Any linear map $T\colon \mathbb{R}^m \to \mathbb{R}^n$. (Question: How about a linear map from $\mathbb{R}^m$ to an NLS?)
(c) $f\colon \mathbb{R} \to \mathbb{R}$ such that $f'(x)$ exists and is bounded. (If you have learnt calculus of several variables, extend this result suitably.)

Ex. 3.4.5. Go through the solution of Ex. 3.2.14 on page 63 and Example 3.2.21 on page 64. Would you like to improve upon the conclusion?

Ex. 3.4.6. Let (x_n) be a sequence in a metric space (X, d). Show that the function $x \mapsto \inf\{d(x, x_n) : n \in \mathbb{N}\}$ is uniformly continuous on X.

Ex. 3.4.7. Show that any uniformly continuous function carries bounded sets to bounded sets.

Ex. 3.4.8. Show that any uniformly continuous function carries Cauchy sequences to Cauchy sequences.

Show that the converse is not true.

Theorem 3.4.9. *Let (X, d) be a metric space. Assume that D is a dense subset of X. Let Y be a complete metric space. Let $f\colon (D, d) \to (Y, d)$ be a uniformly continuous function. There exists a uniformly continuous function $g\colon X \to Y$ such that $g(x) = f(x)$ for all $x \in D$. (The function g is called an extension of f from D to X.)*

Proof. We shall only sketch the proof, as filling in the details will be a very instructive exercise to the reader.

Let $x \in X$. Then by density of D in X, there exists a sequence (x_n) in D such that $x_n \to x$. Note that (x_n) is Cauchy. Since f is uniformly continuous, the sequence $(f(x_n))$ is Cauchy in Y (by Ex. 3.4.8). Since Y is complete, there exists $y \in Y$ such that $f(x_n) \to y$. We set $g(x) = y$. We need to show that $g(x)$ is well-defined in the sense that if (x'_n) in D is convergent to x and if $f(x'_n) \to y'$, then $y = y'$. (Note that this will also show that $g(x) = x$ for $x \in D$, as we may take the constant sequence $(x_n := x)$ convergent to x.)

One then shows that g is, in fact, uniformly continuous on X. $\square$

An instructive application of the theorem is the extension of the meaning of a^r for $r \in \mathbb{Q}$ to a^x for any $x \in \mathbb{R}$ for any fixed $a > 0$. Starting from the existence of n-th roots (Theorem 5.1.22), one assigns a meaning to $a^{m/n}$ where $m, n \in \mathbb{Z}$ and $n \neq 0$. One also verifies the laws of exponents hold. We shall assume these results in the proof of the theorem below.

Theorem 3.4.10. *Fix a positive $a \in \mathbb{R}$ and $N \in \mathbb{N}$. Then the function $r \mapsto a^r$ from $\mathbb{Q} \cap [-N, N]$ to $\mathbb{R}$ is uniformly continuous. Hence it extends to a continuous function from $[-N, N]$ to $\mathbb{R}$. This function is denoted by a^x for $x \in [-N, N]$.*

Proof. We shall give only an outline of the proof. The reader is encouraged to supply the details.

Let $x, x + h \in \mathbb{Q} \cap [-N, N]$. We estimate

$$\left|a^{x+h} - a^x\right| = a^x \left|a^h - 1\right| \leq a^N \left|a^h - 1\right|.$$

If we show that $a^h \to 1$ as $h \to 0$ in $\mathbb{Q}$, then we are through. (Why? How does the uniform continuity follow?) This follows from the fact that $a^{1/n} \to 1$ as $n \to \infty$. (How?)

We also used the fact that $x \mapsto a^x$ is an increasing function on $\mathbb{Q}$. (Where?) Prove this. $\square$

Remark 3.4.11. Since a^x defined on various $[-N, N]$ coincide on their common domain, it follows that we have a function $x \mapsto a^x$ for all $x \in \mathbb{R}$.

Ex. 3.4.12. A function $f\colon (X, d) \to (Y, d)$ is said to be *Lipschitz* if there exists a constant $L > 0$ (called a Lipschitz constant of f) such that for all $x_1, x_2 \in X$, we have

$$d(f(x_1), f(x_2)) \leq L d(x_1, x_2).$$

Show that any Lipschitz continuous function is uniformly continuous.

Ex. 3.4.13. Show that the functions of Ex. 3.4.4-(c) are Lipschitz.

Ex. 3.4.14. Let $T\colon X \to Y$ be a linear map between two NLS. Show that T is continuous iff it is Lipschitz.

Ex. 3.4.15. Let $f\colon \mathbb{R} \to \mathbb{R}$ be differentiable with $|f'(x)| \leq M$. Then f is Lipschitz. *Hint:* Mean value theorem. If you have learnt calculus of several variables, you should extend this result.

3.5 Limit of a Function

Let $f\colon D \subset X \to Y$ be a function. Let a be a point of X, possibly not in D. We wish to assign a meaning to the symbol $\lim_{x\to a, x\in D} f(x) = y$. What we have in mind is that if $x \in D$ is near to a then $f(x)$ is near to y. We first define this notion in the context of a metric space and then extend it to arbitrary topological space. It is intuitively clear that we need points (other than a) of D arbitrarily close to a, that is, a must be a cluster point of D.

Definition 3.5.1. Let (X,d) and (Y,d) be metric spaces. Let $D \subset X$ and assume that $a \in X$ is a cluster point of D. Assume that $f\colon D \setminus \{a\} \to Y$ be given. We say that $\lim_{x\in D; x\to a} f(x)$ exists if there exists $y \in Y$ such that for every given $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x \in B'(a,\delta) \cap D$, we have $d(f(x), y) < \varepsilon$. (See Figure 3.8.) If such a y exists, it is called the limit of f as $x \to a$ and is denoted by $\lim_{x\to a} f(x) = y$.

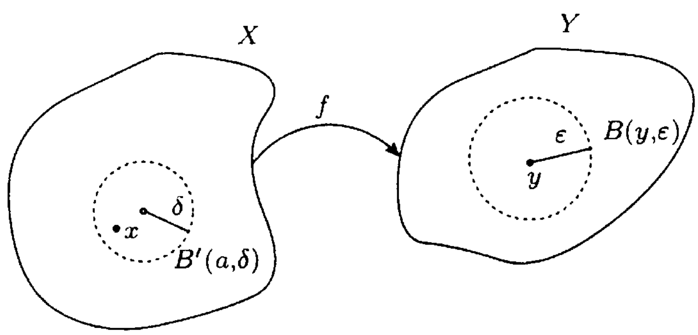

Figure 3.8: $\lim\limits_{x\to a} f(x) = y$

Note that we have used the definite article 'the' and it is justified in the next exercise.

Ex. 3.5.2. Keep the notation of the definition. If there exists $y_1, y_2 \in Y$ such that $\lim_{x\to a} f(x) = y_1$ and $\lim_{x\to a} f(x) = y_2$, then $y_1 = y_2$. *Hint:* You learnt to prove the uniqueness of the limit of a convergent sequence.

Ex. 3.5.3. How do you assign a meaning to $\lim_{x\to a} f(x) = y$ if $f\colon D \subset X \to Y$ is a function between two arbitrary topological spaces? Do you think uniqueness of limit still holds true? What additional condition on (X or on Y) will be needed to ensure uniqueness?

Ex. 3.5.4. Can you reformulate the notion of continuity at a point a in terms of the limit of a functions as $x \to a$?

3.6 Open and closed maps

These are very useful concepts in topology.

Definition 3.6.1. Let X, Y be (metric) spaces. A map $f\colon X \to Y$ is said to be *open* if the image $f(U)$ is open for every open set $U \subset X$. It is said to be *closed* if $f(K)$ is closed for every closed subset $K \subset X$.

Ex. 3.6.2. Show that the projection map $\pi_j\colon \mathbb{R}^n \to \mathbb{R}$ given by $\pi_j(x) = x_j$ is open. It is not closed. *Hint:* Consider $\{(x, y) : x > 0 \text{ and } xy = 1\} \subset \mathbb{R}^2$ and the projection onto the x-axis.

Ex. 3.6.3. Let $X := (\mathbb{R}, d)$ be the usual real line and $Y := (\mathbb{R}, \delta)$ be the set $\mathbb{R}$ with discrete metric δ. Show that the identity map $i\colon X \to Y$ is not continuous but it is open as well as closed. (See Ex. 3.6.2.)

On the other hand, $i\colon Y \to X$ is continuous which is neither open nor closed.

Ex. 3.6.4. A bijection $f\colon X \to Y$ is open iff it is closed.

Ex. 3.6.5. Let A be subset of a (metric) space given the induced topology. When is the inclusion map $a \mapsto a$ of A into X is open? When is it closed?

Ex. 3.6.6. Prove the following:

(a) If $f\colon X \to Y$ is a continuous and onto, then a map $g\colon Y \to Z$ is open if $g \circ f$ is open.

(b) If $g\colon Y \to Z$ is a continuous injection, then a map $f\colon X \to Y$ is open if $g \circ f$ is open.

(c) Do (a) and (b) remain true if we replace 'open' by 'closed'?

Chapter 4

Compactness

4.1 Compact Spaces and their Properties

The basic intuitive idea of a compact subset of a (metric) space is that it is a generalization of a finite set. Most professional mathematicians may subscribe to this view but no author of a book would like to put it in print. But you have it here! Look for instances where this crude intuition helps us conjecture results.

We first introduce the concept of an indexed family of subsets of a set. A thorough knowledge of this will be needed to understand open covers etc.

Let X be any nonempty set. Let I be another nonempty set. A family of subsets of X indexed by the *index set* I is a map $I \to P(X)$ of I into the power set of X, that is, the set of all subsets of X. We denote the image of $i \in I$ by A_i or some such notation. Then the family is denoted by $\{A_i : i \in I\}$. (Compare this with the case of sequences. A sequence in X is a function from $\mathbb{N}$ to X and in practice, we denote it by (x_n) etc.) Let us look at some examples.

Example 4.1.1. Let $X = \mathbb{R}^2$ and $I = [0, \infty)$. For any $r \in I$, we let $C_r := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = r^2\}$ be the circle of radius r with centre at the origin. Here the family is $\{C_r : r \in [0, \infty)\}$.

Example 4.1.2. Let $X = \mathbb{R}$ an $I = \mathbb{Q}^+$, the set of positive rational numbers. For any $r \in \mathbb{Q}$, we let J_r denote the interval $(-r, r)$. Here the family is $\{J_r : r \in \mathbb{Q}^+\}$.

Let a family $\{A_i : i \in I\}$ of subsets of X indexed by I be given. Let $\Lambda \subset I$ be a subset. Then the family $\{A_i : i \in \Lambda\}$ is called a subfamily of the given family $\{A_i : i \in I\}$.

The subset $\cup_{i \in \Lambda} A_i$ of X defined by

$$\cup_{i \in \Lambda} := \{x \in X : x \in A_i \text{ for some } i \text{ in } \Lambda\}$$

is called the union of the subfamily. Let us again look at some examples.

In Example 4.1.1, if we take $\Lambda = [1, 2]$, then the union of the subfamily $\{C_r : r \in [1, 2]\}$ is the annular region $\{(x, y) \in \mathbb{R}^2 : 1 \leq x^2 + y^2 \leq 4\}$. If we take $\Lambda = [0, 1]$, is is the closed disk or the closed ball $B[0, 1]$ in $\mathbb{R}^2$.

In Example 4.1.2, if we take $\Lambda = \mathbb{N}$, then the union of the subfamily $\{J_n : n \in \mathbb{N}\}$ is $\mathbb{R}$. Note that it is also the union of the family $\{J_r : r \in [0, \infty)\}$!

Definition 4.1.3. Let X be a (metric) space and $A \subset X$. A family of subsets $\{U_i : i \in I\}$ is called an *open cover* of A if each U_i is open and $A \subset \cup_i U_i$. (Note that U_i need not be contained in A and they are required to be open in X.)

If $J \subset I$ is a subset of I such that $A \subset \cup_{i \in J} U_i$, we then say $\{U_i : i \in J\}$ is a subcover of the given open cover of A.

If there exists a finite subset $J \subset I$ such that the $A \subset \cup_{i \in J} U_i$, then we say that the given cover admits a finite subcover for A.

Example 4.1.4. Consider the family $\{(1/n, 1) : n \in \mathbb{N}, n \geq 2\}$. We show that this is an open cover of $(0, 1)$. We need to show that $(0, 1) \subset \cup_{n=2}^{\infty} (1/n, 1)$. Let $x \in (0, 1)$. We want to show that $x \in (1/n, 1)$ for some $n \geq 2$. If there exits such an n, then $1/n < x < 1$. Hence $n > 1/x$. By Archimedean property of $\mathbb{R}$, $\mathbb{N}$ is not bounded above in $\mathbb{R}$. Hence $1/x$ is not an upper bound for $\mathbb{N}$. It follows that there exists $N \in \mathbb{N}$ such that $N > 1/x$ or $1/N < x < 1$. Thus the given family is an open cover of $(0, 1)$.

We claim that the given cover admits no finite subcover for $(0, 1)$. Suppose that it does. Then $(0, 1) \subset \cup_{j=1}^{k} (1/n_j, 1)$ for some $n_1, n_2, \ldots, n_k$. Note that $(1/m, 1) \subset (1/n, 1)$ for all $n \geq m$. If we set $N := \max\{n_j : 1 \leq j \leq k\}$, then $\cup_{j=1}^{k} (1/n_j, 1) = (1/N, 1)$. Hence we deduce that $(0, 1) \subset (1/N, 1)$. This is absurd, since $1/2N \in (0, 1)$ but $(1/2N) \notin (1/N, 1)$. Hence the claim follows.

Suppose we add two open intervals, say, J_0, J_1 with the property that $0 \in J_0$ and $1 \in J_1$. Then the collection $\{(1/n, 1) : n \geq 2\} \cup \{J_1, J_2\}$ is an open cover of $[0, 1]$. We claim that this cover admits a finite subcover of $[0, 1]$!

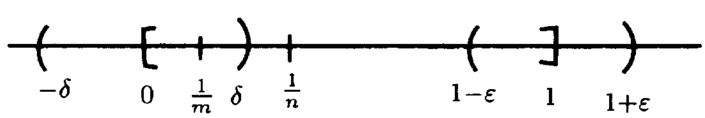

Figure 4.1: Illustration for Example 4.1.4

Since $0 \in J_0$, there exists $\delta > 0$ such that $(-\delta, \delta) \subset J_0$. There exists $m \in \mathbb{N}$ such that $1/m < \delta$, by Archimedean property of $\mathbb{R}$. Similarly, since $1 \in J_1$, there exists $\varepsilon > 0$ such that $(1-\varepsilon, 1+\varepsilon) \subset J_1$. By Archimedean property of $\mathbb{R}$, we can find $n \in \mathbb{N}$ such that $1/n < 1-\varepsilon$. Let $N := \max\{m, n\}$. Then $[0,1] \subset J_0 \cup J_1 \cup (1/N, 1)$. (Verify.)

> Let $x \in [0,1]$. If $x = 0$ or 1, there $x \in J_0 \cup J_1$. If $0 < x < 1$, then either $x \le 1/N$ or $1/N < x < 1$. In the former case, $x \le 1/N \le 1/m < \delta$ and hence $x \in (-\delta, \delta) \subset J_0$. In the later, $x \in (1/N, 1)$.

Thus, the given cover of $[0,1]$ admits a finite subcover of $[0,1]$.

Ex. 4.1.5. Show that each of the following families is an open cover of $(0,1)$:

(1) $\{(0, (n-1)/n) : n \in \mathbb{N}\}$.
(2) $\{(\frac{1}{2} - \delta_n, \frac{1}{2} + \delta_n) : \delta_n := \frac{n-2}{2n}, n \ge 3, n \in \mathbb{N}\}$.

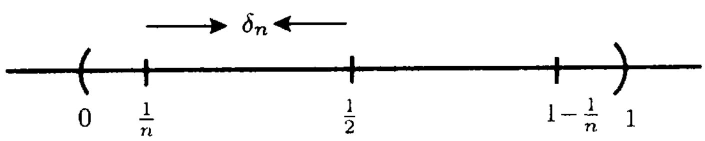

Figure 4.2: Illustration for Ex. 4.1.5

Show that none of these open covers admit a finite subcover.

Add one more open set (or two more open intervals) to each of these covers so that the resulting family is an open cover of $[0,1]$. Show that the latter admits a finite subcover of $[0,1]$. *Hint:* Proceed as in the last example.

Ex. 4.1.6. The family $\{(-x, x) : x \in (0, \infty)\}$ is an open cover of $\mathbb{R}$. Consider $\mathbb{N} \subset (0, \infty)$. Then $\{(-n, n) : n \in \mathbb{N}\}$ is a subcover. Does this admit a finite subcover?

Ex. 4.1.7. Give a 'nontrivial' open cover of an arbitrary metric space.

Ex. 4.1.8. Show that given any open cover of $A = \{1/n : n \in \mathbb{N}\} \cup \{0\}$ (considered as a subset of $\mathbb{R}$) we can find a finite number of elements in the cover such that their union contains A.

Can we assert such an analogous result for the set $B = \{1/n : n \in \mathbb{N}\}$?

Ex. 4.1.9. Let $X = \mathbb{Z}$. Can you think of an open cover of X which does not admit a finite subcover?

Ex. 4.1.10. Let X be an infinite set. Let X be endowed with discrete metric. Give an open cover of X which does not admit a finite subcover.

Definition 4.1.11. A subset $K \subset X$ of a (metric) space is said to be *compact* if every open cover of K admits a finite subcover.

Ex. 4.1.12. Show that any finite subset of a metric space is compact.

Theorem 4.1.13 (Heine-Borel Theorem for $\mathbb{R}$). *Let $J := [a, b]$ be a closed and bounded interval in $\mathbb{R}$. Then, any open cover of $[a, b]$ admits a finite subcover, that is, $[a, b]$ is comapct.*

Proof. Assume that the result is false. Then, there exists an open cover $\{U_i : i \in I\}$ of J which does not admit a finite subcover for J.

The strategy is to bisect J and observe that $\{U_i : i \in I\}$ does not admit a finite subcover for one of the subintervals. Call it J_1. Repeat the argument replacing J by J_1 and so on. We shall have a nested sequence (J_k) such that $\{U_i : i \in I\}$ does not admit a finite subcover for any of J_k's. If $c \in \cap_k J_k$, then $c \in U_\alpha$ for some $\alpha \in I$. One then shows that $J_k \subset U_\alpha$ for all k sufficiently large. Let us now work out the details.

We bisect $J = [a, \frac{a+b}{2}] \cup [\frac{a+b}{2}, b]$. Note that $\{U_i : i \in I\}$ is an open cover for each of these subintervals. We claim that $\{U_i : i \in I\}$ does not admit a finite subcover for any of these subintervals. For, if did, say,

$$[a, \frac{a+b}{2}] \subset U_{i_1} \cup \cdots \cup U_{i_m} \text{ and } [\frac{a+b}{2}, b] \subset U_{j_1} \cup \cdots \cup U_{j_n},$$

then

$$[a, b] \subset (U_{i_1} \cup \cdots \cup U_{i_m}) \cup (U_{j_1} \cup \cdots \cup U_{j_n}).$$

Thus $\{U_i : i \in I\}$ admits a finite subcover for $[a, b]$, a contradiction to our hypothesis. Hence the claim follows.

We select one subinterval for which $\{U_i : i \in I\}$ has no finite subcover. We denote it by $J_1 = [a_1, b_1]$. Note that $a \leq a_1$ (as $a_1 = a$ or $a_1 = (a + b)/2$). We observe the following:

1. $\{U_i : i \in I\}$ does not admit a finite subcover for J_1.
2. The length $\ell(J_1) = b_1 - a_1 = (b - a)/2$.
3. The left end-point a_1 of J_1 is greater than or equal to that of J, that is, $a \leq a_1$.

We repeat the argument replacing J by J_1 as $\{U_i : i \in I\}$ is an open cover of J_1 admitting no finite subcover for J_1. We then obtain a subinterval $J_2 = [a_2, b_2]$ of J_1 such that the following hold:

1. $\{U_i : i \in I\}$ does not admit a finite subcover for J_2.
2. The length $\ell(J_2) = (b_2 - a_2) = (b_1 - a_1)/2 = 2^{-2}(b - a)$.
3. The left end-point a_2 of J_2 is greater than or equal to that of J_1, that is, $a_1 \leq a_2$.

We now proceed recursively. Thus we get a sequence of subintervals (J_k) with the following properties:

1. $\{U_i : i \in I\}$ does not admit a finite subcover for J_k for an $k \in \mathbb{N}$.
2. The length $\ell(J_k) = b_k - a_k = 2^{-k}(b - a)$.

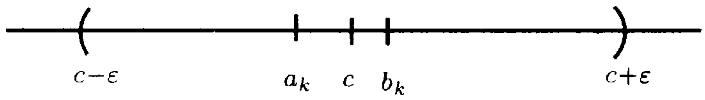

Figure 4.3: Illustration for Thm. 4.1.13

3. The left end-point a_k of J_k is greater than or equal to that of J_{k-1}, that is, $a_{k-1} \leq a_k$.

Consider the sequence (a_k). It is a non-decreasing sequence of real numbers, that is, $a_{k+1} \geq a_k$ for all $k \in \mathbb{N}$. Also, it is bounded. For, all $a_k \in [a,b]$. Hence, from a standard result from real analysis, the sequence (a_k) converges to $c := \sup\{a_k\}$. We observe that $c \in [a,b]$. For, c is an upper bound for all a_k and in particular, $c \geq a_1 \geq a$. Also, c is the least upper bound for the set $\{a_k : k \in \mathbb{N}\}$ and b is its upper bound. Hence $a \leq c \leq b$.

We claim that $b_k \to c$. For,

$$|b_k - c| \leq |b_k - a_k| + |a_k - c| = 2^{-k}(b-a) + |a_k - c| \to 0,$$ [1]

as $k \to \infty$.

Since $c \in [a,b]$ and $\{U_i : i \in I\}$ is an open cover of $[a,b]$, there exists $\alpha \in I$ such that $c \in U_\alpha$. Since U_α is open (this is the only place where we use the fact that the sets in $\{U_i : i \in I\}$ are open!), there exists $\varepsilon > 0$ such that $(c-\varepsilon, c+\varepsilon) \subset U_\alpha$. Since $a_k \to c$ and $b_k \to c$, there exist positive integers k_1 and k_2 such that

$$a_k \in (c-\varepsilon, c+\varepsilon) \text{ for } k \geq k_1 \text{ and } b_k \in (c-\varepsilon, c+\varepsilon) \text{ for } k \geq k_2.$$

Hence if $k \geq k_0 := \max\{k_1, k_2\}$, then $a_k, b_k \in (c-\varepsilon, c+\varepsilon)$. Therefore,

$$[a_k, b_k] \subset (c-\varepsilon, c+\varepsilon) \subset U_\alpha.$$

This implies that $\{U_i : i \in I\}$ admits a finite subcover for J_k (if $k \geq k_0$). This contradicts our assumption on J_k. This contradiction shows that out original assumption that $\{U_i : i \in I\}$ does not admit a finite subcover for J is wrong. Hence the theorem follows. $\square$

Ex. 4.1.14. Find all compact subsets of a discrete metric space.

While Theorem 4.1.13 gave an example of an uncountable compact set, the next exercise gives an example of a compact set which is infinitely countable.

[1] We used the fact that $2^{-k} \to 0$. It follows from the observation that $k \leq 2^k$ for all $k \in \mathbb{N}$. This is proved by induction. Consequently, $2^{-k} < 1/k$.

Ex. 4.1.15. Show that $\{1/n : n \in \mathbb{N}\} \cup \{0\}$ is compact in $\mathbb{R}$. Can you generalize this? (in a metric space?)

Theorem 4.1.16. *Any compact subset of a metric space is closed and bounded.*

Proof. Let $K \subset X$ be compact. Fix any $x_0 \in X$. Consider the open cover $\{B(x_0, n) : n \in \mathbb{N}\}$. This admits a finite subcover for K, say, $\{B(x_0, n_j) : 1 \leq j \leq n\}$. Since $B(x_0, m) \subset B(x_0, n)$ for $n \geq m$, it follows that $K \subset B(x_0, N)$ for $N := \max\{n_j : 1 \leq j \leq n\}$. Thus, K is bounded.

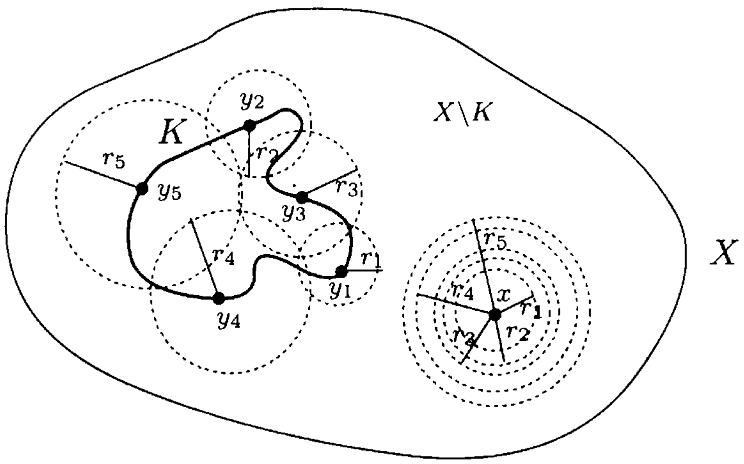

Figure 4.4: Compact subset is closed

To show that K is closed, let $x \notin K$. For each $y \in K$, using the Hausdorff property of X, we can find r_y such that $B(x, r_y) \cap B(y, r_y) = \emptyset$. The collection $\{B(y, r_y) : y \in K\}$ is obviously an open cover of K. Since K is compact, we have a finite subcover, say $B(y_i, r_i)$, $1 \leq i \leq n$, where $r_i := r_{y_i}$. (See Figure 4.4.) Clearly, $B := \cap_i B(x, r_i)$ is an open set containing x. We claim that $B \cap K = \emptyset$. Let, if possible, $x \in B \cap K$. Since $K \subset \cup_{i=1}^{n} B(y_i, r_i)$, there exists some j such that $x \in B(y_j, r_j)$. Since $B \subset B(x_i, r_i)$ for all i, it follows that $x \in B(x_j, r_j) \cap B(y_j, r_j)$, a contradiction.

Thus for every $x \notin K$, we have found an open set $U \ni x$ such that $U \subset X \setminus K$, that is, $X \setminus K$ is open. $\square$

Remark 4.1.17. The last theorem has a partial analogue in the context of Hausdorff spaces. A compact subset of a Hausdorff space is closed. The reader can adapt the proof given above for metric spaces. See Figure 4.5.

Ex. 4.1.18. Let A, B be compact subsets of a (metric) space X. Is $A \cup B$ compact?

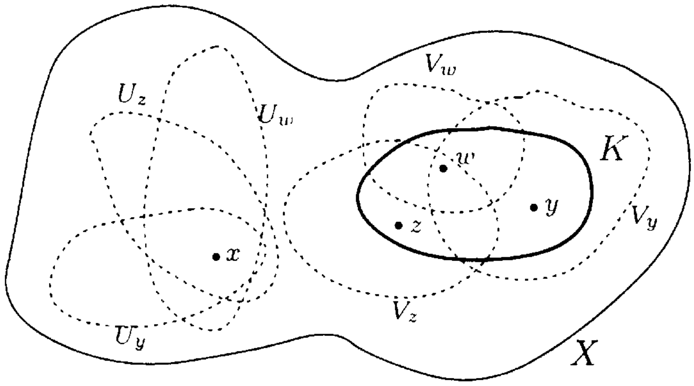

Figure 4.5: Compact subset is closed in a Hausdorff space

The assumptions are the same as above but we insist that X is a metric space. Is $A \cap B$ compact?

Ex. 4.1.19. Consider $G := (0, \infty)$ with the metric induced from $\mathbb{R}$. Note that G is a group under multiplication. Which subgroups of G are compact subsets of the metric space G?

Ex. 4.1.20. Show that any closed subset of a compact set in a (metric) space X is compact.

Ex. 4.1.21. Show that any closed and bounded subset of $\mathbb{R}$ is compact. *Hint:* Ex. 4.1.20 and Theorem 4.1.13 on page 84.

Ex. 4.1.22. A subset of $\mathbb{R}$ is compact if and only if it is closed and bounded.

We shall show later that a similar result is true in $\mathbb{R}^n$ for any n (Theorem 4.1.30). In a general metric space, this need not be true. See the next few exercises.

Ex. 4.1.23. Let X be infinite and d be the discrete metric on X. Show that X is bounded and closed but not compact.

Ex. 4.1.24. The set $(-\sqrt{2}, \sqrt{2}) \cap \mathbb{Q}$ is a closed and bounded subset in $\mathbb{Q}$, but not compact.

Example 4.1.25. We show that the closed unit ball $B[0,1]$ in ℓ^2 is not compact. Consider $e_n \in \ell^2$ defined by $e_n(k) = \delta_{nk}$,[2] for $n, k \in \mathbb{N}$. Thus, e_n is the sequence all of whose terms are zero except the n-th term. Clearly, $e_n \in B[0,1]$ and $d(e_m, e_n) = \sqrt{2}$ if $m \neq n$. We claim that the open cover

$$\{B(x, 1/2) : x \in B[0,1]\}$$

[2] δ_{rs} is the Kronecker delta symbol defined as $\delta_{rs} = 1$ if $r = s$ and 0 otherwise.

of $B[0,1]$ admits no finite subcover. For, if $\{B(x_j, 1/2) : 1 \leq j \leq n\}$ is a finite subcover, then there exists j $(1 \leq j \leq n)$ and $m \neq n$ such that $e_m, e_n \in B(x_j, 1/2)$. In particular, $d(e_m, e_n) \leq d(e_m, x_j) + d(x_j, e_n) < 1$, a contradiction.

Ex. 4.1.26. Show that the unit ball in $(C[0,1], \| \ \|_\infty)$ is not compact. *Hint:* The strategy is to adapt the argument of the last example. For this purpose, we need to construct a sequence (f_n) of functions in $B[0,1]$ such that $d(f_m, f_n) = 1$ if $m \neq n$. For each $n \in \mathbb{N}$, we consider the function which takes the value 0 at $t = 0, \frac{1}{n+1}, \frac{1}{n-1}, 1$ and the value 1 at $\frac{1}{n}$ and at other points it is extended by 'linearity'. See Figure 4.6.

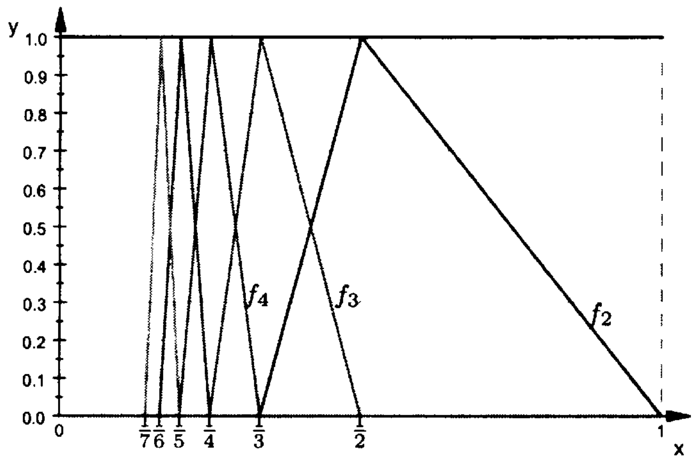

Figure 4.6: Graphs of f_n's in Exercise 4.1.26

To be very concrete, consider the sequence of functions defined as follows:

$$f_n(t) := \begin{cases} 0, & 0 \leq t \leq \frac{1}{n+1}; \\ n(n+1)\left(t - \frac{1}{n+1}\right), & \frac{1}{n+1} \leq t \leq \frac{1}{n}; \\ n(1-n)\left(t - \frac{1}{n-1}\right), & \frac{1}{n} \leq t \leq \frac{1}{n-1}; \\ 0, & \frac{1}{n-1} \leq t \leq 1. \end{cases}$$

Ex. 4.1.27. Let $a, b, c, d \in \mathbb{R}$ be such that $b - a = d - c$. Let $S := [a,b] \times [c,d]$ be the square in $\mathbb{R}^2$. The vertices of S are (a,c), (b,c), (b,d) and (a,d). We call the point (a,c) as the bottom left vertex of S. The pair of midpoints of its opposite sides are given by

$$([a+b]/2, c), ([a+b]/2, d) \text{ and } (a, [c+d]/2), (b, [c+d]/2]).$$

By joining the midpoints of opposite sides, we get four smaller squares. Observe that if (a_1, c_1) is the bottom left vertex of any of these squares, we have $a \leq a_1$ and $c \leq c_1$. See Figure 4.7.

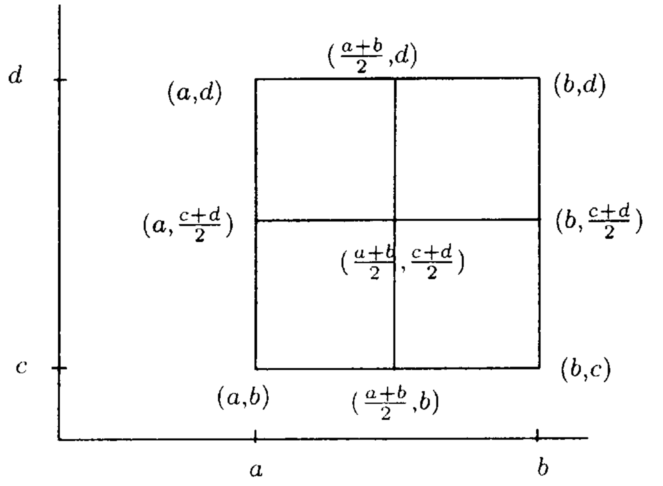

Figure 4.7: Illustration for Exercise 4.1.27

Lemma 4.1.28. *The square $S := [-R, R] \times [-R, R]$ is a compact subset of $\mathbb{R}^2$.*

Proof. We mimic the argument of Theorem 4.1.13.

Suppose that S is not compact. Then there is an open cover $\{U_i : i \in I\}$ of which there is no finite subcover of S. Let us divide the square S into four smaller squares by joining the pairs of midpoints of opposite sides. (See Exercise 4.1.27 above.) One of these square will not have a finite subcover from the given cover. For, otherwise, each of these four squares will have finite subcover so that the union of these subcovers will be a cover for S. Thus, S itself will admit a finite subcover. Choose one such smaller square and call it S_1. Note that the length of its sides is R and that if (a_1, c_1) is the bottom left vertex of S_1, then $a_1 \geq a_0 = -R$ and $c_1 \geq c_0 = -R$. We repeat the argument by subdividing S_1 into four squares and choosing one of the smaller squares which does not admit a finite subcover of $\{U_i\}$. Call this smaller square as S_2. (See Figure 4.8.) Note that the length of its sides is $R/2$ and that if (a_2, c_2) is the bottom left vertex of S_2, then $a_1 \leq a_2$ and $c_1 \leq c_2$.

Proceeding recursively, we have a sequence of squares S_n such that S_n dose not admit a finite subcover and the length of sides of S_n is $2^{-n+1}R$ and its bottom left vertex (a_n, c_n) is such that $a_{n-1} \leq a_n$ and $c_{n-1} \leq c_n$. Thus we have two sequences of real numbers (a_n) and (c_n). They are bounded and monotone. Hence there exist real numbers a and c such that $a_n \to a$ and $c_n \to c$. It follows that $(a_n, c_n) \to (a, c) \in \mathbb{R}^2$. Since S is closed, we infer that $(a, c) \in S$. Hence there is U_{i_0} in the open cover such that $(a, c) \in U_{i_0}$. Since U_{i_0} is open there exists an $r > 0$ such that $B((a, c), r) \subset U_{i_0}$.

Choose $n \in \mathbb{N}$ with the following properties:

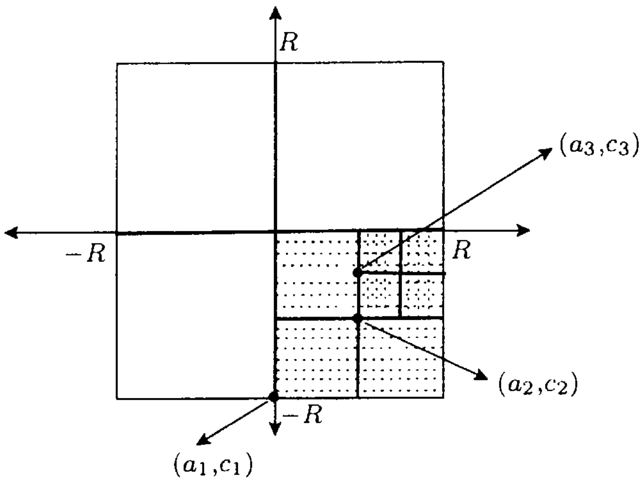

Figure 4.8: $[-R, R] \times [-R, R]$ is compact

(1) $\operatorname{diam} S_n = 2^{-n+1}\sqrt{2}R < r/2$ and
(2) $d((a,c),(a_n,c_n)) < r/2$.
We then have, for any $(x,y) \in S_n$,

$$\begin{aligned} d((a,c),(x,y)) &\leq d((a,c),(a_n,c_n)) + d((a_n,c_n),(x,y)) \\ &< r/2 + 2^{-n+1}\sqrt{2}R < r. \end{aligned}$$

Thus $S_n \subset B((a,c),r) \subset U_{i_0}$. But then $\{U_{i_0}\}$ is a finite subcover for S_n, contradicting our choice of S_k's. Therefore, our assumption that S is not compact is not tenable. □

Ex. 4.1.29. Note that the proof above generalizes to $\mathbb{R}^n$ including the case when $n = 1$.

Theorem 4.1.30 (Heine-Borel Theorem). *A subset of $\mathbb{R}^n$ is compact iff it is closed and bounded.*

Proof. This is an easy corollary of Theorem 4.1.16 on page 86, Ex. 4.1.20 on page 87, Lemma 4.1.28 on page 89 and Ex. 4.1.29. □

Ex. 4.1.31. Which of the following are compact? Justify your answers.
(a) The unit sphere $S^{n-1} := \{x \in \mathbb{R}^n : \|x\| = 1\}$.
(b) The hyperbola $x^2 - y^2 = 1$ in $\mathbb{R}^2$.
(c) The parabola $y^2 = x$ in $\mathbb{R}^2$.
(d) The ellipse $(x^2/a^2) + (y^2/b^2) = 1$.
(e) A 'conic section' in $\mathbb{R}^2$ given by a second degree polynomial in x and y.
(f) The set of points $x \in \mathbb{R}^n$ such that $x_1^2 + 2x_2^2 + \cdots + nx_n^2 \leq (n+1)^2$.
(g) A nonzero vector subspace of a (nonzero) NLS.

Ex. 4.1.32. Is the set $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = 1\}$ compact in $\mathbb{R}^3$?

Ex. 4.1.33. Show that the set $O(n)$ of all orthogonal matrices is a compact subset of $M(n, \mathbb{R})$. (Recall that if $A = (C_1, \ldots, C_n)$ is orthogonal with C_k as the k-th column, then the dot product of the column vectors satisfy the equations: $\langle C_k, C_j \rangle = \delta_{jk}$.)

Ex. 4.1.34. Are $GL(n, \mathbb{R})$ and $SL(n, \mathbb{R})$ compact? How about the set of nilpotent matrices?

Ex. 4.1.35. Which of the vector subspaces of an NLS are compact?

4.2 Continuous Functions on Compact Spaces

In the next proposition, note the presence of (metric) for X and its absence for Y!

Proposition 4.2.1. *Any continuous function f from a compact (metric) space (X, d) to another metric space (Y, d) is bounded, that is, $f(X)$ is a bounded subset of Y*

Proof. Understand the proof when $Y = \mathbb{R}$ or $Y = \mathbb{C}$.

Fix any $y \in Y$. For each $n \in \mathbb{N}$, the set $B(y, n)$ is an open set in Y and hence

$$U_n := f^{-1}(B(y, n)) = \{x \in X : d(f(x), y) < n\}$$

is open. Note also that $U_n \subset U_{n+1}$. The collection $\{U_n : n \in \mathbb{N}\}$ is an open cover of the compact space X. If $\{U_{n_i} : 1 \leq i \leq m\}$ is a finite subcover and if $N := \max\{n_i\}$, then $X = U_N$, that is, $f(X) \subset B(y, N)$. Hence $f(X)$ is bounded. □

Ex. 4.2.2. Assume that the metric space (X, d) is not compact. Show that there exists $f: X \to \mathbb{R}$ which is continuous but not bounded.

Theorem 4.2.3. *Any continuous function from a compact (metric) space to $\mathbb{R}$ attains its bounds.*

More precisely, let X be compact and $f: X \to \mathbb{R}$ be continuous. Let $M := M := \sup\{f(x) : x \in X\}$ and $m := \inf\{f(x) : x \in X\}$. Then there exist $x_1, x_2 \in X$ such that $f(x_1) = M$ and $f(x_2) = m$.

Proof. By the last proposition, there exists an $\alpha \in \mathbb{R}$ such that $|f(x)| \leq \alpha$ for all $x \in X$. Hence m and M exist as real numbers. We shall prove the existence of x_1 by contradiction. If there is no $p \in X$ such that $f(p) = M$, then the sets

$$U_n := \{x \in X; f(x) < M - \frac{1}{n}\}$$

form an open cover of X. (For, if $x \in X$, then $f(x) < M$ and hence by Archimedean property of $\mathbb{R}$, there exists $n \in \mathbb{N}$ such that $n > \frac{1}{M-f(x)}$ whence $f(x) < M - \frac{1}{n}$ or $x \in U_n$.) Note that $U_n \subset U_{n+1}$ for all n. As in the last proof, we conclude that $X = U_N$ for some N. But this leads to the contradiction: $\sup\{f(x) : x \in X\} \leq M - \frac{1}{N}$! □

Ex. 4.2.4. Let $f\colon X \to (0, \infty)$ be a continuous function on a compact space X. Show that there exists $\varepsilon > 0$ such that $f(x) \geq \varepsilon$ for all $x \in X$.

Ex. 4.2.5. Show that the continuous image of a compact (metric) space is compact.

Hence conclude that compactness is a topological property: if X and Y are homeomorphic, and if X is compact, so is Y.

Ex. 4.2.6. A standard proof of Proposition 4.2.1 and Theorem 4.2.3 uses Ex. 4.2.5, Theorem 4.1.16 and Ex. 2.2.11. Work out this proof and compare this with the ones given above.

Ex. 4.2.7. Let $K \subset \mathbb{R}^n$ be with the property that any real valued continuous function on K is bounded. Show that K is compact. *Hint:* Heine-Borel Theorem.

Theorem 4.2.8. *Any continuous function from a compact metric space to any other metric space is uniformly continuous.*

Proof. Let $f\colon (X, d) \to (Y, d)$ be continuous. Assume that X is compact. We need to prove that f is uniformly continuous on X.

A naive attempt would runs as follows. For a given $\varepsilon > 0$, for each x, there exists $\delta_x > 0$ by continuity of f at x. Since X is compact, the open cover $\{B(x, \delta_x) : x \in X\}$ admits a finite subcover, say, $\{B(x_i, \delta_i) : 1 \leq i \leq n\}$ where $\delta_i = \delta_{x_i}$. One may be tempted to believe that if we set $\delta := \min\{\delta_i : 1 \leq i \leq n\}$, it might work. See where the problem is. Once you arrive at a complete proof, you may refer to the proof below.

We now modify the argument and complete the proof. Given $\varepsilon > 0$, by the continuity of f at x, there exists an $r_x > 0$ such that

$$d(x, y) < r_x \implies d(f(x), f(y)) < \varepsilon/2.$$

Instead of the open cover $\{B(x, r_x) : x \in X\}$, we consider the open cover $\{B(x, r_x/2) : x \in X\}$ and apply compactness. Let $\{B(x_i, r_i/2) :$

$1 \leq i \leq n\}$ be a finite subcover. (Here $r_i = r_{x_i}$, $1 \leq i \leq n$.) Let $\delta := \min\{r_i/2 : 1 \leq i \leq n\}$. Let $x, y \in X$ be such that $d(x, y) < \delta$. Now $x \in B(x_i, r_i/2)$ for some i. Since $d(x, y) < \delta$, we see that

$$d(y, x_i) \leq d(y, x) + d(x, x_i)) \leq \delta + r_i/2 < r_i.$$

Thus $y \in B(x_i, r_i)$. It follows that

$$d(f(x), f(y)) \leq d(f(x), f(x_i)) + d(f(x_i) + f(y)) < \varepsilon/2 + \varepsilon/2,$$

by our choice of r_i. Thus f is uniformly continuous.

Another proof is indicated in Ex. 4.3.17 and a third is given in Example 4.3.5. □

Ex. 4.2.9. Let $f: (0, 1) \to \mathbb{R}$ be continuous, monotone and bonded. Show that f is uniformly continuous on $(0, 1)$. *Hint:* $\lim_{x \to 0_+} f(x)$ and $\lim_{x \to 1_-} f(x)$ exist so that f extends as a continuous function on $[0, 1]$.

Ex. 4.2.10. Let $f: (X, d) \to (Y, d)$ be a map of metric spaces. Assume that f is *locally Lipschitz*, that is, for each $x \in X$, there exists an open ball B_x containing x and a constant $L_x > 0$ such that

$$d(f(x_1), f(x_2)) \leq L_x d(x_1, x_2) \text{ for all } x_1, x_2 \in B_x.$$

If X is compact, then f is Lipschitz. *Hint:* With the notation of the exercise, select a finite cover $\{B_{x_i} : 1 \leq i \leq n\}$. Let L_i be the Lipshitz constant corresponding to the $B_i := B_{x_i}$. The continuous function

$$d: X \times X \setminus (\cup_{i=1}^{n} B_i \times B_i) \to \mathbb{R}$$

is positive and attains a minimum δ. Choose M such that $d(f(x), f(y)) \leq M\delta$. Then $L := \max\{M, L_1, \ldots, L_n\}$ is a Lipschitz constant for f on X.

Ex. 4.2.11. We say that a function $f: [a, b] \to \mathbb{R}$ is *linear* if it is of the form $f(t) = \alpha t + \beta$ for some $\alpha, \beta \in \mathbb{R}$. Show that f is determined by its values at two (distinct) points in $[a, b]$. More precisely, arrive at an expression for a linear f such that $f(x_1) = y_1$ and $f(x_2) = y_2$, $x_1, x_2 \in [a, b]$.

Definition 4.2.12. A function $g: [0, 1] \to \mathbb{R}$ is said to be piecewise linear if we can find a partition $0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1$ of the interval $[0, 1]$ such that g is a linear on each of the subintervals (x_j, x_{j+1}). It is continuous iff it is linear on each of the closed subintervals. (Justify this.)

Proposition 4.2.13 (Density of Piecewise Linear Functions). *Given a continuous function $f\colon [0,1] \to \mathbb{R}$, and $\varepsilon > 0$, there exists a piecewise linear continuous function g on $[0,1]$ such that $|f(x) - g(x)| < \varepsilon$ for $x \in [0,1]$. That is, the set of piecewise linear continuous functions on $[0,1]$ is dense in $(C[0,1], \| \ \|_\infty)$.*

Proof. Let $\varepsilon > 0$ be given. Since f is uniformly continuous on $[0,1]$, there exists $\delta > 0$ such that whenever $|x - y| < \delta$, we have $|f(x) - f(y)| < \varepsilon/2$. Choose a positive integer N so large that $1/N < \delta$. We partition the interval $[0,1]$ into N equal parts: $0 < 1/N < 2/N < \cdots < (N-1)/N < 1$. Let $x_i = i/N$ for $0 \le i \le N$.

Let $g\colon [0,1] \to \mathbb{R}$ be the function that is linear on each of the subintervals $[x_i, x_{i+1}]$ such that $g(x_i) = f(x_i)$ and $g(x_{i+1}) = f(x_{i+1})$. (See Figure 4.9. Write down g explicitly.)

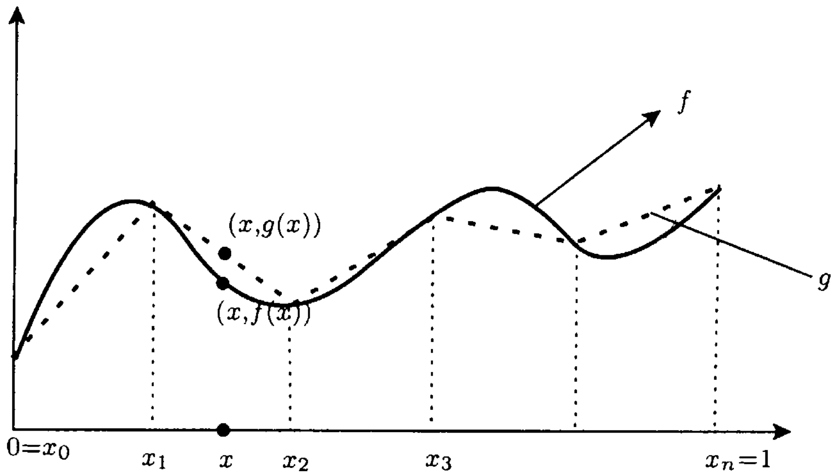

Figure 4.9: Density of piecewise linear functions

Let $x \in [x_i, x_{i+1}]$, for some $0 \le i \le N-1$. Then $g(x)$ lies between the values $g(x_i) = f(x_i)$ and $g(x_{i+1}) = f(x_{i+1})$. Hence

$$|f(x_i) - g(x)| \le |f(x_i) - f(x_{i+1})| < \varepsilon/2.$$

Also, $|f(x) - f(x_i)| < \varepsilon/2$, it follows that $|f(x) - g(x)| < \varepsilon$. ☐

Ex. 4.2.5 can be used to show that certain spaces are not homeomorphic.

Ex. 4.2.14. Show that a circle in $\mathbb{R}^2$ is not homeomorphic to a parabola or a hyperbola.

Show also that $\mathbb{R}$ is not homeomorphic to an ellipse.

Ex. 4.2.15 (A very useful fact). Let X, Y be metric spaces. Assume that X is compact. Let $f\colon X \to Y$ be a bijective continuous map.

Show that the inverse map $f^{-1}: Y \to X$ is continuous, that is, f is a homeomorphism. *Hint:* Observe that the image of a closed set under f is closed. Why? See Theorem 4.1.16 and Remark 4.1.17.

This has the following extension in the context of topological spaces. A bijective continuous map f from a compact space to a Hausdorff space is continuous. (You need Remark 4.1.17.)

4.3 Characterization of Compact Metric Spaces

Lemma 4.3.1 (Lebesgue Covering Lemma). *Let (X, d) be a compact metric space. Let $\{U_i\}$ be an open cover of X. Then there is a $\delta > 0$ such that if $A \subset X$ with diameter* $\operatorname{diam}(A) < \delta$, *then there is an i such that $A \subset U_i$.*

Remark 4.3.2. If δ is as in the theorem and $0 < \delta' \leq \delta$, then δ' also has the required property. Any δ of the theorem is called a Lebesgue number of the covering $\{U_i\}$.

Ex. 4.3.3. Let $X = (0, 1)$ and $U_n = (1/n, 1)$. Does a Lebesgue number exist for this cover?

Ex. 4.3.4. Let $U_0 := (-1/10, 1/10)$ and $U_a := (a/2, 2)$ for $0 < a \leq 1$. Show that $U_0 \cup \{U_a : 0 < a \leq 1\}$ is an open cover of $[0, 1]$. Find a Lebesgue number of this cover.

We now go to a proof of the Lebesgue covering lemma.

Proof. For $x \in X$, there is an $i(x)$ such that $x \in U_{i(x)}$ and an $r(x) > 0$ such that $B(x, 2r(x)) \subset U_{i(x)}$. (Why?) There exist finitely many x_k, $1 \leq k \leq n$ such that $X = \cup_k B(x_k, r_k)$ where $r_k := r(x_k)$. Let δ be any positive real such that $\delta < \min\{r_k\}$. Let A be any subset with $\operatorname{diam}(A) < \delta$. Let $a \in A$. Then $a \in B(x_k, r_k)$ for some k. Let $x \in A$ be arbitrary. Then $d(x, x_k) \leq d(x, a) + d(a, x_k) < \delta + r_k < 2r_k$. Thus, $A \subset B(x_k, 2r_k) \subset U_{i(x_k)}$. ☐

Example 4.3.5. One may use Lebesgue covering lemma to give an alternative proof of Theorem 4.2.8. Given $\varepsilon > 0$, by the continuity of f at $x \in X$, there exists $\delta_x > 0$ such that if $d(x, x') < \delta_x$, then $d(f(x), f(x')) < \varepsilon/2$. Now the family $\{B(x, \delta_x) : x \in X\}$ is an open cover of the compact metric space X. Let δ be a Lebesgue number of the covering. Let $x_1, x_2 \in X$ be such that $d(x_1, x_2) < \delta$. Then $\operatorname{diam}\{x_1, x_2\} < \delta$ and hence there exists a member $B(x, \delta_x)$ of the covering with $x_1, x_2 \in B(x, \delta_x)$. It follows that

$$d(f(x_1), f(x_2)) \leq d(f(x_1), f(x)) + d(f(x), f(x_2)) < \varepsilon/2 + \varepsilon/2.$$

Therefore, f is uniformly continuous on X.

Compare this proof with the one given in Theorem 4.2.8.

Definition 4.3.6. Let A be a subset of a metric space. We say that A is *totally bounded* if for every $\varepsilon > 0$, we can find a finite number of points x_i, $1 \leq i \leq n$, such that $A \subset \cup_{i=1}^n B(x_i, \varepsilon)$.

Clearly, this is a back-door entry of compactness! Do you see why?

We say that a subset $A \subset X$ is an ε-net if $d_A(x) < \varepsilon$ for any $x \in X$. Thus X is totally bounded iff there exists a finite ε-net for every $\varepsilon > 0$.

Ex. 4.3.7. Show that any compact space is totally bounded.

Ex. 4.3.8. Show that any bounded subset of $\mathbb{R}$ is totally bounded.

Can you generalize this to $\mathbb{R}^n$? to any metric space?

Ex. 4.3.9. Show that if B is totally bounded and $A \subset B$, then A is totally bounded.

Ex. 4.3.10. If A is totally bounded subset of a metric space, show that its closure $\overline{A}$ is also totally bounded.

Ex. 4.3.11. Show that a metric space (X, d) is totally bounded iff each infinite subset of X contains distinct points that arbitrarily close to each other.

Ex. 4.3.12. Let (X, d) be a metric space which is not totally bounded. Prove that there exists a sequence (x_n) in X and a positive η such that $d(x_m, x_n) \geq \eta$ whenever $m \neq n$.

Ex. 4.3.13. Show that a subset $D \subset X$ is dense iff it is ε-net for every $\varepsilon > 0$.

The following result gives three characterizations of compact metric spaces. Each of these is quite useful. One use is, of course, to decide whether a given metric space is compact or not. The other is to apply them to derive some results about compact metric spaces. We shall illustrate their uses below. Most often for theoretical purpose, characterization (2) of the theorem seem to be efficient. See Theorem 4.4.8 and Ex. 4.4.10. Condition (4) is quite useful when dealing with problems, where we have some idea of the convergence in the given metric space. See Ex. 4.3.18–Ex. 4.3.23. Only experience will teach us which characterization is useful or most expedient in a given context.

Theorem 4.3.14 (Characterization of Compact Metric Spaces). *For a metric space (X, d), the following are equivalent:*

(1) *X is compact: every open cover has a finite subcover.*

(2) *X is complete and totally bounded.*

(3) *Every infinite set has a cluster point.*

(4) *Every sequence has a convergent subsequence.*

Proof. **(1) $\implies$ (2):** Let (X, d) be compact. Given $\varepsilon > 0$, the family $\{B(x, \varepsilon) \mid x \in X\}$ is an open cover of X. Let $\{B(x_i, \varepsilon) \mid 1 \le i \le n\}$ be a finite subcover. Hence X is totally bounded.

Now let (x_n) be a Cauchy sequence in X. Then for every $k \in \mathbb{N}$ there exits n_k such that $d(x_n, x_{n_k}) < 1/k$ for all $n > n_k$. Let

$$U_k := \left\{ x \in X \;\middle|\; d(x, x_{n_k}) > \frac{1}{k} \right\}.$$

Then U_k is open.[3] Now $x_n \notin U_k$ for $n > n_k$. Hence no finite subcover of U_k's cover X: For, if they did, say, $X = \cup_{i=1}^m U_i$, we take $n > \max\{n_1, \dots, n_m\}$. Then $x_n \notin U_k$ for any k with $1 \le k \le m$. This implies that $\{U_k\}$ cannot cover X. Thus there exists $x \in X \setminus \cup_{k=1}^\infty U_k$. But then $d(x, x_{n_k}) < \frac{1}{k}$. Hence $x_{n_k} \to x$. Since (x_n) is Cauchy we see that x_n also converges to $\lim_k x_{n_k}$. Thus X is complete. We have thus shown (1) implies (2).

(2) $\implies$ (3): Let E be an infinite subset of X. Let F_n be a finite subset of X such that $X = \cup_{x \in F_n} B(x, \frac{1}{n})$. Then for $n = 1$ there exists $x_1 \in F_1$ such that $E \cap B(x_1, 1)$ is infinite. As a second step, replace E by $E \cap B(x_1, 1)$ and F_1 by F_2. The infinite set $E \cap B(x_1, 1)$ is contained in the finite union $\cup_{x \in F_2} B(x, 1/2)$. Hence there exists $x_2 \in F_2$ such that

$$E \cap B(x_1, 1) \cap B(x_2, 1/2) \text{ is infinite.}$$

Inductively choose $x_n \in F_n$ such that $E \cap (\cap_{k=1}^n B(x_k, \frac{1}{k}))$ is infinite. Since there is $a \in E \cap B(x_m, \frac{1}{m}) \cap B(x_n, \frac{1}{n})$ we see that

$$d(x_m, x_n) \le d(x_m, a) + d(a, x_n) < \frac{1}{m} + \frac{1}{n} < \frac{2}{m} \text{ for } m < n.$$

Thus (x_n) is Cauchy. Since X is complete x_n converges to some $x \in X$. Also $d(x, x_n) \le 2/n$ for all n.[4] Thus $B(x, 3/n)$ includes $B(x_n, \frac{1}{n})$ which includes infinitely many elements of E. Thus x is a cluster point of E. Hence (3) is proved.

(3) $\implies$ (4): If (x_n) is a sequence in X we let $x(\mathbb{N}) := \{x_n \mid n \in \mathbb{N}\}$ be its image. If this set is finite then (4) trivially follows.

> Let $x(\mathbb{N}) = \{z_1, \dots, z_m\}$. Consider the subset $S_k := x^{-1}(z_k)$. Since $\mathbb{N} = \cup_{k=1}^m S_k$, at least one of the subsets, say, S_1 is infinite. Using well-ordering principle of $\mathbb{N}$, we can write S_1 as $\{n_1 < n_2 < \cdots\}$. Then the subsequence (x_{n_k}) is a constant sequence z_1 and hence converges to z_1.

[3] It is the complement of the closed set $B[x_{n_k}, 1/n_k]$. Or, if $y \in U_k$ and $\delta := d(x_{n_k}, y) - \frac{1}{k}$ then $B(y, \delta) \subseteq U_k$. A third way is to use the continuity of $x \mapsto d(X, x_{n_k})$.

[4] Note that $d(x, x_n) := \lim_{m \to \infty} d(x_m, x_n)$ by the continuity of $x \mapsto d(x, x_n)$. Since for $m > n$, $d(x_m, x_n) < 2/n$, the claim follows.

So assume that $\{x_n \mid n \in \mathbb{N}\}$ is infinite. Let x be a cluster point of this set. Then there exist elements x_{n_k} such that $d(x, x_{n_k}) < 1/k$ for all k. Thus $x_{n_k} \to x$ and (4) is thereby proved.

(4) $\implies$ (1): Let $\{U_\alpha\}$ be an open cover of X. For $x \in X$, let

$$r_x := \sup\{r \in \mathbb{R} \mid B(x,r) \subseteq U_\alpha \text{ for some } \alpha\}.$$

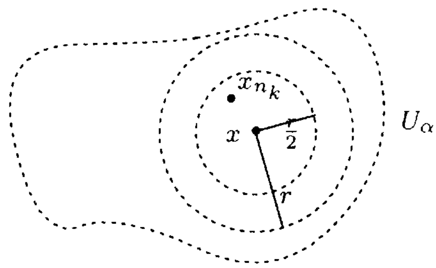

Figure 4.10: (4) $\implies$ (1) of Theorem 4.3.14

We claim that $\varepsilon := \inf\{r_x \mid x \in X\} > 0$. If not, there is a sequence (x_n) such that $r_{x_n} \to 0$. But (x_n) has a convergent subsequence, say, $x_{n_k} \to x$. Now $x \in U_\alpha$ for some α and hence there is an $r > 0$ such that $B(x,r) \subset U_\alpha$. For k large enough $d(x, x_{n_k}) < \frac{r}{2}$ so that $r_{x_{n_k}} > \frac{r}{2}$ for all sufficiently large k – a contradiction. Hence the claim is proved.

Let $\varepsilon := \inf\{r_x \mid x \in X\}$. Choose any $x_1 \in X$. Inductively choose x_n such that $x_n \notin \cup_{k=1}^{n-1} B(x_i, \varepsilon/2)$. We cannot do this for all n. For otherwise, (x_n) will not have a convergent subsequence since $d(x_n, x_m) > \frac{\varepsilon}{2}$ for all $m \neq n$. Hence $X = \cup_{k=1}^{N} B(x_k, \frac{\varepsilon}{2})$ for some N. But then for each k there is an α_k such that $B(x_k, \frac{\varepsilon}{2}) \subset U_{\alpha_k}$. Hence $X = \cup_{k=1}^{N} U_{\alpha_k}$. Thus $\{U_\alpha\}$ has a finite subcover or X is compact. ☐

Ex. 4.3.15. Consider $B[0,1]$, the closed unit ball in $C[0,1]$ under the sup norm. Show that it is not compact. *Hint:* Can you think of a sequence which has no convergent subsequence?

Remark 4.3.16. We give a second proof of Lebesgue covering lemma to illustrate the typical use of characterization (4) of Theorem 4.3.14.

Suppose that Lebesgue covering lemma is false. Then, for any $\delta = 1/n$, there is a subset A_n with diameter less than $1/n$ and such that it is not a subset of U_i for any i. Choose any $x_n \in A_n$. Then the sequence (x_n) has a convergent subsequence (x_{n_k}) such that $x_{n_k} \to p$ in X. Let $p \in U_i$. Let $r > 0$ be such that $B(p, 2r) \subset U_i$. Choose k so large that $x_{n_k} \in B(p,r)$ and $1/n_k < r$. Now if $a \in A_{n_k}$ is any element, then, $d(a,p) \leq d(a, x_{n_k}) + d(x_{n_k}, p) < 2r$. That is, $A_{n_k} \subset B(p, 2r) \subset U_i$, contradicting our assumption on the A_n's. ☐

Ex. 4.3.17. Give an alternative proof of Theorem 4.2.8 along the following lines. Assume that f is not uniformly continuous. Therefore there exist some $\varepsilon > 0$ and two sequences (x_n) and (y_n) such that $d(x_n, y_n) \to 0$ but $d(f(x_n), d(y_n)) \geq \varepsilon$. *Hint:* Go through Remark 4.3.16

Many of the exercises below admit more than one approach. We indicate them in some of the exercises, so that the reader can try different approaches.

Ex. 4.3.18. The product metric space of two compact metric spaces is compact. *Hint:* Use the 4th characterization in Theorem 4.3.14 of a compact metric space.

Given two topological spaces X, Y, one can define the product topology on the Cartesian product $X \times Y$ and show that the product space of compact spaces is again compact. We do not get into this.

Ex. 4.3.19. Given A, B two compact subsets of a metric space such that $A \cap B = \emptyset$. Show that $d(A, B) > 0$. In fact, show that there exist $a \in A$ and $b \in B$ such that $d(A, B) = d(a, b)$. *Hint:* You can use the continuity of d_A and Theorem 4.2.3. Or use the last exercise and Theorem 4.2.3. A third approach would be to use the 4th characterization in Theorem 4.3.14 of a compact metric space.

Ex. 4.3.20. Let C be closed and K be compact in $\mathbb{R}^n$. Assume that $C \cap K = \emptyset$. Show that $d(K, C) > 0$. *Hint:* Most efficient solution would use Ex. 4.2.4. As an alternative, we also ask you to use the 4th characterization of a compact metric space.

Ex. 4.3.21. Let C be closed and K be compact in $\mathbb{R}^n$. Show that $K + C$ is closed.

Ex. 4.3.22. Let A, B be compact subsets of $\mathbb{R}^n$. Show that their sum $A + B$ is compact. *Hint:* Vector addition is continuous. Alternatively, use the 4th characterization of a compact metric space.

Ex. 4.3.23. Redo Ex. 4.2.10 using the 4th characterization in Theorem 4.3.14 of a compact metric space. (This approach is the standard one!)

A very instructive and useful exercise involving some typical compactness argument is the following result.

Theorem 4.3.24. *Any two norms on $\mathbb{R}^n$ are equivalent, that is, the topologies induced by these norms are the same.*

Proof. It suffices to show that a given norm on $\mathbb{R}^n$ is equivalent to the Euclidean norm. (Why?)

Let η denote a norm on $\mathbb{R}^n$. We continue to denote the Euclidean norm by $\| \ \|$. The norms are equivalent iff there exist positive constants C_1, C_2 such that

$$C_1 \|x\| \leq \eta(x) \leq C_2 \|x\| \text{ for all } x \in \mathbb{R}^n. \tag{4.1}$$

Let $\{e_k : 1 \leq k \leq n\}$ be the standard basis of $\mathbb{R}^n$. Then we have for any $x = (x_1, \ldots, x_n) = \sum_{k=1}^n x_k e_k$,

$$\begin{aligned} \eta(x) \equiv \eta\left(\sum_{k=1}^n x_k e_k\right) &\leq \sum_{k=1}^n |x_k| \, \eta(e_k) \\ &\leq M \sum_{k=1}^n |x_k| \\ &\leq M \sum_{k=1}^n \|x\| \\ &= Mn \|x\|, \end{aligned} \tag{4.2}$$

where $M := \max\{\eta(e_k) : 1 \leq k \leq n\}$. Thus the right most inequality in (4.1) is obtained with $C_2 := Mn$. (An aside: One can, in fact, improve this constant to $M\sqrt{n}$, by applying the Cauchy-Schwarz inequality to the sum $\sum_{j=1}^n |x_j| \, y_j$ where $y_j = 1$ for all $1 \leq j \leq n$.)

To get the left most inequality of (4.1), we make some preliminary observation. If $\|x\| = 1$, what it means is that

$$\eta(x) \geq C_1 \text{ for all } x \in S := \{x \in \mathbb{R}^n : \|x\| = 1\}.$$

If $x \in S$, then $\|x\| = 1$ and hence $x \neq 0$. It follows that the map satisfies: $x \mapsto \eta(x) > 0$ for $x \in S$. What we want to claim is that it is bounded below by a positive constant C_1. This triggers our memory. We have seen something similar in Ex. 4.2.4. So what we need to show is that $\eta : S \to \mathbb{R}$ is a continuous function on the compact subset $S \subset (\mathbb{R}^n, \| \ \|)$. But we have done it already. The inequality (4.2) shows that η is Lipschitz continuous from $(S, \| \ \|)$ to $\mathbb{R}$:

$$\left|\eta(x) - \eta(y)\right| \leq \eta(x - y) \leq Mn \|x - y\|,$$

by Ex. 1.1.39 and (4.2).

Hence the function η restricted to S attains the minimum value, say C_1 at some point $x_0 \in S$. Hence $\eta(x) \geq C_1$ for all $x \in S$. Consequently, $\eta(\frac{x}{\|x\|}) \geq C_1$ for all nonzero $x \in \mathbb{R}^n$. It follows that $\eta(x) \geq C_1 \|x\|$ for all $x \in \mathbb{R}^n$. $\square$

Ex. 4.3.25. Adapt the above proof to show that any two norms on a finite dimensional (real/complex) vector space are equivalent.

4.4 Arzela-Ascoli Theorem

The theorem of the title gives an immensely useful criterion of compactness of subsets of $C(X)$ where X is a compact metric space and $C(X)$ is given the sup norm metric.

Definition 4.4.1. Let X and Y be metric spaces and $a \in X$. A family $\mathcal{A}$ of functions from X to Y is said to be *equicontinuous* at a if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$d(x, a) < \delta \Rightarrow d(f(x), f(a)) < \varepsilon \qquad \text{for all } f \in \mathcal{A}.$$

We say that $\mathcal{A}$ is equicontinuous on X if it is equicontinuous at each point of X.

A slightly stronger notion is introduced in the following

Definition 4.4.2. Keep the notation of the last definition. We say that $\mathcal{A}$ is *uniformly equicontinuous* on X if for any given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$d(x_1, x_2) < \delta \Rightarrow d(f(x_1), f(x_2)) < \varepsilon \qquad \text{for all } f \in \mathcal{A}.$$

Ex. 4.4.3. Any member of a uniformly equicontinuous family is uniformly continuous.

Ex. 4.4.4. Let $f\colon X \to Y$ be any continuous function between metric spaces. Then
(i) $\mathcal{A} := \{f\}$ is equicontinuous on X iff ....
(ii) $\mathcal{A} := \{f\}$ is uniformly equicontinuous on X iff ....

The following two exercises give two of the most important ways uniformly equicontinuous families arise.

Ex. 4.4.5. Let X be a compact metric space. Let $F\colon X \times X \to Z$ be continuous. Let $f_y(x) := F(x, y)$. Show that $\mathcal{A} := \{f_y : y \in X\}$ is uniformly equicontinuous.

Ex. 4.4.6. Let $X \subset \mathbb{R}^n$ be convex and open. Let $\mathcal{A}$ be a family of differentiable functions from X to $\mathbb{R}^m$. Assume that there exists $M > 0$ such that $\|Df(x)\| \leq M$ for all $f \in \mathcal{A}$ and for all $x \in X$. Show that $\mathcal{A}$ is uniformly equicontinuous.

If X is a compact metric space, then there is no distinction between equicontinuous family and uniformly equicontinuous family. This is the content of the next lemma.

Lemma 4.4.7. *Let X be a compact metric space and $\mathcal{A}$ be a family of equicontinuous functions from X to another metric space Y. Then $\mathcal{A}$ is uniformly equicontinuous family.*

Proof. Proceed exactly as in the proof of Theorem 4.2.8. Observe that you can choose the δ_x's independent of $f \in \mathcal{A}$ thanks to the fact that $\mathcal{A}$ is equicontinuous at $x \in X$. We leave the details to the reader. □

Theorem 4.4.8 (Arzela-Ascoli Theorem). *Let X be a compact metric space. Let $C(X, \mathbb{K})$ be given the sup norm metric. ($\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$.) Then a set $\mathcal{B} \subset C(X)$ is compact iff $\mathcal{B}$ is bounded, closed and equicontinuous.*

Proof. Assume that $\mathcal{B}$ is compact. Then $\mathcal{B}$ is closed and totally bounded since $C(X)$ is a complete metric space. Given $\varepsilon > 0$ there exists $f_i \in \mathcal{B}$ for $1 \leq i \leq n$ such that $\mathcal{B} \subset \cup_i B(f_i, \varepsilon)$. Let δ_i be chosen by the uniform continuity of f_i for the given ε. Let δ be the minimum of the δ_i's. This δ does the job.

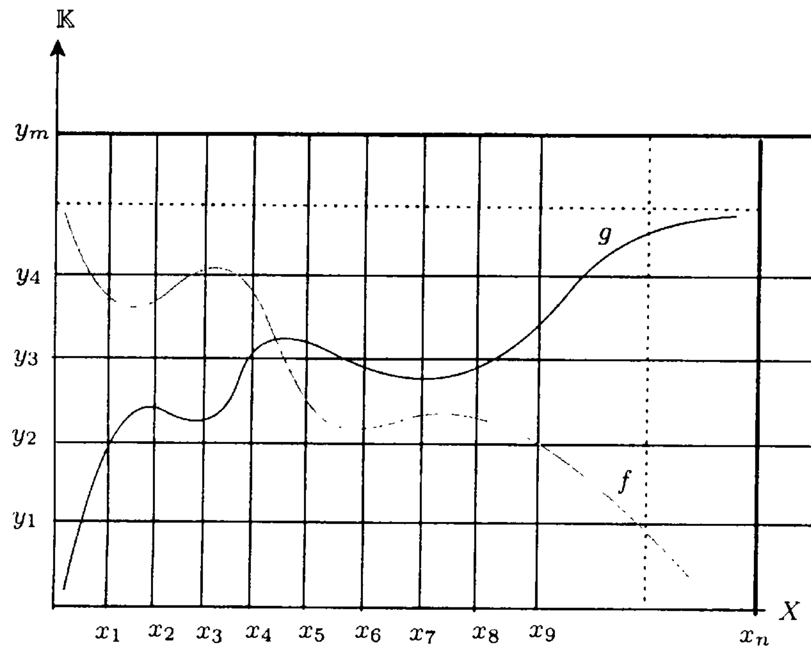

Figure 4.11: Arzela-Ascoli

Now assume that $\mathcal{B}$ is bounded, closed and equicontinuous. Since X is compact, $\mathcal{B}$ is uniformly equicontinuous. Since $(C(X), \| \ \|_\infty)$ is complete, $\mathcal{B}$ is complete. So it is enough if we show that $\mathcal{B}$ is totally bounded.

Let $\varepsilon > 0$ be given. We plan to show that there exists a 4ε-net for $\mathcal{B}$. Let M be such that $|f(x)| \leq M$ for all $x \in X$ and $f \in \mathcal{B}$. Using the uniform equicontinuity of $\mathcal{B}$ we get a δ with the following property:

$$\left|f(x) - f(x')\right| < \varepsilon \text{ whenever } d(x, x') < \delta \text{ for all } f \in \mathcal{B}.$$

We can find $x_i \in X$, $1 \leq i \leq m$ such that $X = \cup_{i=1}^{m} B(x_i, \delta)$. Since $B[0, M]$ the closed ball of radius M centred at 0 in $\mathbb{K}$ is compact and hence totally bounded, we can find $y_j \in B[0, M]$, $1 \leq j \leq n$ such that $B[0, M] \subset \cup B(y_j, \varepsilon)$. Let $A := \{\alpha \colon \{x_i\} \to \{y_j\}\}$ be the set of all functions α from the set $\{x_i : 1 \leq i \leq m\}$ to the set $\{y_j : 1 \leq j \leq n\}$. Then $|A| = n^m$. For $\alpha \in A$, let

$$U_\alpha := \{f \in \mathcal{B} : |f(x_i) - \alpha(i)| \leq \varepsilon\}.$$

(Refer to Fig. 4.11 on page 102.)

We claim that the diameter of U_α is at most 4ε. We need to show that $\|f - g\|_\infty \leq \varepsilon$, that is, to show that $|f(x) - g(x)| \leq 4\varepsilon$ for al $x \in X$ and $f, g \in U_\alpha$. Let $x \in X$ be given. Then $x \in B(x_i, \delta)$ for some $1 \leq i \leq m$. It follows that

$$\begin{aligned} |f(x) - f(x_i)| &< \varepsilon && (4.3) \\ |g(x) - g(x_i)| &< \varepsilon. && (4.4) \end{aligned}$$

Also, since $f, g \in U_\alpha$, we have

$$\begin{aligned} |f(x_i) - \alpha(x_i)| &\leq \varepsilon \\ |g(x_i) - \alpha(x_i)| &\leq \varepsilon. \end{aligned}$$

It follows that

$$|f(x_i) - g(x_i)| \leq 2\varepsilon. \qquad (4.5)$$

We therefore deduce from (4.3)–(4.5) that

$$|f(x) - g(x)| \leq |f(x) - f(x_i)| + |f(x_i) - g(x_i)| + |g(x_i) - g(x)| \leq 4\varepsilon.$$

Hence we conclude that the diameter of U_α is at most 4ε.

We show that the union of U_α's is $\mathcal{B}$. Let $f \in \mathcal{B}$ be given. For any i $(1 \leq i \leq n)$, let $j := j(i)$ $(1 \leq j(i) \leq n)$ be chosen so that $f(x_i) \in B(y_j, \varepsilon)$. Define $\alpha(x_i) = y_{j(i)}$. Then, $|f(x_i) - \alpha(x_i)| < \varepsilon$. That is, $f \in U_\alpha$. □

Remark 4.4.9. In the above proof, we showed that $\mathcal{B} \subset \cup_{\alpha \in A} U_\alpha$. To show total boundedness of $\mathcal{B}$, we need to exhibit ε-net of open balls. This can be done, if we fix $f_\alpha \in U_\alpha$, then $U_\alpha \subset B(f_\alpha, 5\varepsilon)$, for any $\alpha \in A$.

Ex. 4.4.10. Prove that a set $A \subseteq \ell_1$ is compact iff (i) A is bounded and (ii) given $\varepsilon > 0$, there exists n_0 such that $\sum_{k=n}^{\infty} |x_k| < \varepsilon$ for all $n \geq n_0$ and $x \in A$. *Hint:* To prove the sufficiency, you need only show that A is totally bounded. Given $\varepsilon > 0$, for n_0 as in the condition, apply Heine-Borel to the finite dimensional space $\{(x_1, \ldots, x_{n_0}, 0, 0, \ldots)\}$.

Ex. 4.4.11. Let X, Y be metric spaces. Assume that Y is compact. Let $f: X \to Y$ be a function. The graph Graph(f) is the subset

$$\text{Graph}(f) := \{(x, f(x)) : x \in X\} \subset X \times Y.$$

Show that f is continuous iff the graph of f is a closed subset of $X \times Y$ under the product metric.

Ex. 4.4.12 (Finite Intersection Property and Compactness). Let X be a set. We say that a collection $\mathcal{A}$ of (nonempty) subsets of X has *finite intersection property* (f.i.p., in short) if every finite family $A_1, \ldots, A_n$ of elements in $\mathcal{A}$ has a nonempty intersection.

Prove the following: A (metric) space is compact iff every family of closed sets with f.i.p. has nonempty intersection. *Hint:* Start with an open cover $\mathcal{U}$ which does not admit a finite subcover. Look at $\{X \setminus U : U \in \mathcal{U}\}$.

Ex. 4.4.13. Show that any compact metric space has a countable dense subset. Can you improve upon this result?

Ex. 4.4.14. Let X be a compact space. Assume that (A_n) is a sequence of non-empty closed sets in X such that $A_{n+1} \subset A_n$. Show that $\cap A_n \neq \emptyset$.

Compare this with Ex. 4.4.12.

Ex. 4.4.15. Let X be a compact metric space. Let $f: X \to X$ be continuous. Show that there exists a nonempty subset A of X such that $f(A) = A$. *Hint:* Consider $A_1 = f(X)$ and $A_{n+1} = f(A_n)$ for $n \geq 1$

Ex. 4.4.16. Let (X, d) be a compact metric space. Let $f: X \to X$ be such that $d(f(x), f(y)) < d(x, y)$ for all $x, y \in X$ with $x \neq y$. Show that f has a fixed point, that is, there exists $x_0 \in X$ such that $f(x_0) = x_0$. Is the fixed point unique? *Hint:* Consider the function $x \mapsto d(x, f(x))$. What contradiction does it lead to if $\inf\{d(x, f(x)) : x \in X\} > 0$?

Ex. 4.4.17. Let (X, d) be a compact metric space. Let $f: X \to X$ be such that $d(f(x), f(y)) = d(x, y)$ for all $x, y \in X$. Show that f is onto. *Hint:* Fix $y \in X$ and $x_1 \in X$. Define $x_n = f(x_{n-1})$. Observe that $d(x_n, x_{n+k}) = d(y, x_n)$.

Ex. 4.4.18. Let $f: X \to Y$ be a continuous map between metric spaces. Assume that K_n is a nonempty compact subset of X and that $K_{n+1} \subset K_n$ for each $n \in \mathbb{N}$. Let $K := \cap_n K_n$. Show that $f(K) = \cap_n f(K_n)$. *Hint:* For the non-trivial part, consider the compact (why?) sets $f^{-1}(y) \cap K_n$ where $y \in \cap_n f(K_n)$.

Ex. 4.4.19. Show that any open cover of the unit circle in $\mathbb{R}^2$ is a cover of an annulus, i.e., if $\{U_i : i \in I\}$ is an open cover of $\{(x, y) \in \mathbb{R}^2 : x^2+y^2 = 1\}$, then it is an open cover of an annulus $1-\delta < \|(x, y)\| < 1+\delta$ for some $\delta > 0$. *Question:* Draw pictures. Are you convinced of the truth?

Ex. 4.4.20. Show that the Hilbert cube (Ex. 1.2.63) is compact. *Hint:* Show that it is complete and totally bounded. Or, you may prove sequential compactness by a diagonal argument. I personally prefer the first, as it is more instructive.

Chapter 5

Connectedness

We say that a (metric) space is connected if it is a 'single piece.' This is a very difficult notion to be formulated precisely. If we look at $\mathbb{R} \setminus \{0\}$, we would think of it consisting of two pieces, namely, one of positive numbers and the other of negative numbers. Similarly, if we consider an ellipse or a parabola, it is in a single space while a hyperbola has two distinct pieces. If we remove a single point from a circle, it still remains as a single piece. We start with a definition which is unintuitive but with examples and further exploration see that it captures our intuitive ideas.

5.1 Connected Spaces

Definition 5.1.1. A (metric) space X is said to be *connected* if the only sets which are both open and closed in X are $\emptyset$ and the full space X, when X is a metric space.

A subset A of a (metric) space X is said to be connected if A is a connected space when considered as a (metric) space with the induced (or subspace) topology. More explicitly, this amounts to saying that (A, δ) is connected, where δ is the restriction of the metric d on X to A.

Ex. 5.1.2. $\mathbb{R}$ is connected (Ex. 2.2.12) while $\mathbb{R}^*$, the subset of nonzero real numbers is not connected.

Ex. 5.1.3. Show that a (metric) space X is not connected iff there exist two disjoint proper non-empty subsets A and B such that A and B are both open and closed in X and $X = A \cup B$. In such a case, we say that the pair (A, B) is a *disconnection* of X.

Ex. 5.1.4. Let X be a set such that $|X| \geq 2$ with discrete metric. Show that X is not connected.

Ex. 5.1.5. When is a finite subset of a metric space connected?

The following result is the most important characterization of connected spaces.

Theorem 5.1.6. *A topological space X is connected iff every continuous function $f\colon X \to \{\pm 1\}$ is a constant function.*

A subset A of X, endowed with the subspace topology, is connected iff every continuous function $f\colon A \to \{\pm 1\}$ is a constant function

Proof. Let X be a connected space and $f\colon X \to \{\pm 1\}$ a continuous function. We want to show that f is a constant function. If f is non-constant, then it is onto. Let $A = f^{-1}(1)$ and $B = f^{-1}(-1)$. Then A and B are disjoint non-empty subsets of X such that A and B are both open and closed subsets of X and $X = A \cup B$.(Why?). This is a contradiction. Therefore f is constant.

Conversely, let us assume that X is not connected. Therefore there exist two disjoint proper non-empty subsets A and B in X such that A and B are both open and closed in X and $X = A \cup B$. Now we define a map $f\colon X \to \{\pm 1\}$ as

$$f(x) = \begin{cases} 1 & \text{if } x \in A \\ -1 & \text{if } x \in B. \end{cases}$$

Then $f\colon X \to \{\pm 1\}$ is a continuous non-constant function. (Why?) This completes the proof. □

Proposition 5.1.7. *The interval $[a, b] \subset \mathbb{R}$ is connected.*

Proof. If we assume the intermediate value theorem from real analysis, we can give a short and elegant proof. If J is an interval and if $f\colon J \to \{\pm 1\}$ is an onto continuous function, then there exist $x, y \in J$ such that $f(x) = -1$ and $f(y) = 1$. By the intermediate value theorem, there exists z between x and y such that $f(z) = 0$, a contradiction to our assumption that f takes only the values ± 1. Hence no such f exists and hence J is connected.

In stead, we may adapt the argument used to prove the intermediate value theorem to show directly that any interval $[a, b]$ is connected and deduce the general result form this.

Assume that $[a, b]$ is not connected. We then can write $[a, b] = U \cup V$ where U and V are nonempty proper open subsets of $[a, b]$ with $U \cap V = \emptyset$. Without loss of generality assume that $a \in U$. We intend to show that $U = [a, b]$ so that $V = \emptyset$.

Consider $E := \{x \in [a, b] : [a, x] \subset U\}$. Since $a \in U$ and U is open there exists an $\varepsilon > 0$ such that $[a, \varepsilon) \subset U$. Hence $[a, \varepsilon/2] \subset U$ or $a + \varepsilon/2 \in E$ so that $E \neq \emptyset$. E is clearly bounded above by b. Thus by the LUB axiom there exists a real number $c \in \mathbb{R}$ which is $\sup E$. Note that $a \leq c \leq b$.

We claim that $c \in E$. For each $n \in N$, $c - 1/n$ is not an upper bound for E. We can therefore find $x_n \in E$ such that $c - 1/n < x_n \leq c$. Clearly $\lim x_n = c$. Since $x_n \in E$, $x_n \in U$. Since U is closed in $[a, b]$ (with respect to the subspace topology) and $c \in [a, b]$, we see that $c = \lim x_n \in U$. Now $[a, c) = \cup[a, c - 1/n] \subseteq \cup[a, x_n]$. As each of $[a, x_n] \subset U$ we see that $[a, c) \subset U$. This along with the fact that $c \in U$ allows us to conclude that $[a, c] \subset U$ and hence $c \in E$.

We now show that $c = b$. This will complete the proof. Since $c \in U$ and U is open there exists an (relatively) open subset containing c lying in U. If $c < b$, then there exists an $N \in \mathbb{N}$ such that $(c - 1/N, c + 1/N) \subset U$. This means that the set $[a, c + 1/2N] \subset [a, c] \cup (c - 1/N, c + 1/N) \subset U$. Thus $c + 1/2N \in E$. This contradicts the fact that $c = \sup E$. Therefore our assumption that $c < b$ is wrong. Thus $c = b$. This completes the proof of connectedness of $[a, b]$. □

The next theorems list some important facts about connected spaces, which are frequently used. All of these facts follow easily from Theorem 5.1.6 and hence the reader should attempt to prove them on his own.

Theorem 5.1.8. (1) *Let X be a (metric) space. Let A and B be two connected subsets of X such that $A \cap B \neq \emptyset$. Then $A \cup B$ is connected.*
(2) *Let A be a connected subset of a (metric) space X. Let $A \subset B \subset \overline{A}$. Then B is connected.*
(3) *Let $\{A_i : i \in I\}$ be a collection of connected subsets of a (metric) space X with the property that for all $i, j \in I$ we have $A_i \cap A_j \neq \emptyset$. Then $A := \cup_i A_i$ is connected.*

Proof. We strongly recommend the reader to supply proofs on his own.

We prove (1). Let $c \in A \cap B$. Let $f\colon A \cap B \to \{\pm 1\}$ be continuous. Since A is connected, f is a constant on A. So, $f(a) = f(c)$ for all $a \in A$. Similarly f is a constant on B and so $f(b) = f(c)$ for all $b \in B$. We have thus shown any continuous function from $A \cup B$ to $\{\pm 1\}$ is a constant and hence $A \cup B$ is connected.

Now to prove (2), let $f\colon B \to \{\pm 1\}$ be a continuous function. Since A is connected, f is a constant, say 1, on A. Let $b \in B$. Since f is continuous at b, given the open set $\{f(b)\}$ in $\{\pm 1\}$, there exists an open set $U \ni b$ such that $f(U) \subset \{f(b)\}$. Since $b \in B \subset \overline{A}$, b is a limit point of A. In particular, there exists $a \in A \cap U$. So, we find that $f(a) \in \{f(b)\}$ or $f(b) = f(a) = 1$. Thus f is a constant on B.

If you read the proofs of (1) and (2), it is time that you proved (3). □

Ex. 5.1.9. Let X be a (metric) space such that given any two points $x, y \in X$ there exists connected set A such that $x, y \in A$. Then X is connected.

Theorem 5.1.10. *Let X be a connected (metric) space and $g\colon X \to Y$ be a continuous map. Then $g(X)$ is connected.*

Proof. We offer two proofs. The first proof uses our criterion. Let $f\colon g(X) \to \{\pm 1\}$ be a continuous function on $f(X)$. Since $f \circ g\colon X \to \{\pm 1\}$ is continuous and X is connected, it follows that $f \circ g$ is a constant on X. Hence f is a constant on $g(X)$. Therefore $g(X)$ is connected.

Assume that $g(X)$ is not connected. Then there exists nonempty proper subset $V \subset g(X)$ which is both open and closed in $g(X)$. Since f is continuous, $g^{-1}(V)$ and $g^{-1}(g(X) \setminus V)$ are both nonempty, closed and open in X. This contradicts the hypothesis that X is connected. $\square$

Remark 5.1.11. As a consequence of the theorem, we see that connectedness is a topological property.

Ex. 5.1.12. Show that the circle $\{(x, y) \in \mathbb{R}^2 : x^2+y^2 = 1\}$ is connected.

Ex. 5.1.13. Show that the following subsets of $\mathbb{R}^2$ are not connected:
(1) $\{(x, y) \in \mathbb{R}^2 : xy \neq 0\}$.
(2) $\{(x, y) \in \mathbb{R}^2 : x^2 - y^2 = 1\}$.
(3) $\{(x, y) \in \mathbb{R}^2 : x \in \mathbb{Q} \text{ and } y \notin \mathbb{Q}\}$.

Ex. 5.1.14. Show that the set $GL(2, \mathbb{R})$ is not connected.

Ex. 5.1.15. The $O(n, \mathbb{R})$ of orthogonal matrices of order n is not connected.

Ex. 5.1.16. Show that the set $SO(2, \mathbb{R}) := \{A \in O(2, \mathbb{R}) : \det A = 1\}$ is connected. *Hint:* Write down all elements of $SO(2, \mathbb{R})$ explicitly.

Ex. 5.1.17. Let X, Y be metric spaces. Assume that X is connected and that $f\colon X \to Y$ is continuous. Show that the graph

$$\Gamma_f := \{(x, y) \in X \times Y : y = f(x), x \in X\}$$

is a connected subset of $X \times Y$ (with respect to the product metric).

Theorem 5.1.18 (Connected Subsets of $\mathbb{R}$). *A set $J \subseteq \mathbb{R}$ is connected iff J is an interval.*

Proof. If $J \subset \mathbb{R}$ is not an interval, then there exist $x, y \in J$ and a z between x and y such that $z \notin J$. Consider the function $f\colon \mathbb{R} \setminus \{z\} \to$

$\{\pm 1\}$ defined by setting $f(s) = -1$ if $s < z$ and $f(s) = 1$ if $s > z$. Then f is a continuous function from J onto $\{\pm 1\}$. Hence J is not connected.

To prove the converse, let J be any nonempty interval. Let $x, y \in J$. Without loss of generality, assume that $x < y$. Since J is an interval, $[x, y] \in J$. By Proposition 5.1.7, $[x, y]$ is a connected set. Thus, J has the property that any two of its points lie in a connected set. Hence J is connected by Ex. 5.1.9. $\square$

Corollary 5.1.19 (Intermediate Value Theorem). *Let $f\colon [a, b] \to \mathbb{R}$ be a continuous function. Assume that y is a point between $f(a)$ and $f(b)$, that is, either $f(a) \le y \le f(b)$ or $f(b) \le y \le f(a)$ holds. Then there exists $x \in [a, b]$ such that $f(x) = y$.*

Proof. This admits an elementary proof in real analysis. However, we show the role of connectedness using some of our recent results.

By Proposition 5.1.7, the interval $[a, b]$ is connected. Since f is continuous on $[a, b]$, the image $f([a, b])$ is connected by Theorem 5.1.10. Any connected subset of $\mathbb{R}$ is an interval (Theorem 5.1.18) and hence $f([a, b])$ is an interval, say, J. Now $f(a), f(b) \in J$. By the definition of an interval, the point $y \in J$, that is, $y \in f([a, b])$. Therefore we conclude that there exists $x \in [a, b]$ such that $f(x) = y$. $\square$

Ex. 5.1.20. Let A be a nonempty connected subset of $\mathbb{R}$. Assume that every point of A is rational. What can you conclude?

Ex. 5.1.21. Find all continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ which take irrational values only at rational points and not at irrational points.

We now give two neat applications of the intermediate value theorem.

Theorem 5.1.22 (Existence of n-th roots). *Let $\alpha \in [0, \infty)$ and $n \in \mathbb{N}$. Then there exists a unique $x \in [0, \infty)$ such that $x^n = \alpha$.*

Proof. The case when $\alpha = 0$ is clear. So we assume that $\alpha > 0$.

Consider the continuous function $f\colon [0, \infty) \to [0, \infty)$ given by $f(t) = t^n$. By Archimedean property of $\mathbb{R}$, there exists $N \in \mathbb{N}$ such that $N > \alpha$. We have $0 = f(0) < \alpha$ and $f(N) = N^n \ge N > \alpha$. So, applying the intermediate value theorem to the restriction of f to the interval $[0, N]$, we see that there exists $x \in [0, N]$ such that $x^n = \alpha$.

Uniqueness is easy and left to the reader. $\square$

Theorem 5.1.23. *Any polynomial with real coefficients and of odd degree has a real root. That is, if $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, then $a_j \in \mathbb{R}$ for $0 \le j \le n$, $a_n \ne 0$ and n is odd, then there exists $\alpha \in \mathbb{R}$ such that $p(\alpha) = 0$.*

Proof. We may assume that $a_n = 1$ and prove the result. (Why?) For $x \neq 0$, we write

$$p(x) = x^n q(x) \quad \text{where } q(x) := 1 + \frac{a_{n-1}}{x} + \frac{a_{n-2}}{x^2} + \cdots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n}.$$

For $|x| > 1$, we have

$$\begin{aligned} |q(x) - 1| &\leq \frac{|a_{n-1}|}{|x|} + \frac{|a_{n-2}|}{|x|^2} + \cdots + \frac{|a_1|}{|x|^{n-1}} + \frac{|a_0|}{|x|^n} \\ &\leq \frac{|a_{n-1}|}{|x|} + \frac{|a_{n-2}|}{|x|} + \cdots + \frac{|a_1|}{|x|} + \frac{|a_0|}{|x|} \\ &= \frac{A}{|x|}, \end{aligned}$$

where $A := \sum_{j=0}^{n-1} |a_j|$. Thus, if $|x| > \max\{1, 2A\}$, then $|q(x) - 1| < 1/2$. Hence $q(x) > 0$ for such x. If we now choose any $\beta > \max\{1, 2A\}$, then $q(\beta)$ and $q(-\beta)$ are both positive. But then $p(\beta) > 0$ and $p(-\beta) < 0$. Hence by intermediate value theorem there exists $\alpha \in [-\beta, \beta]$ such that $p(\alpha) = 0$. □

Ex. 5.1.24. Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function. "Identify" $f([a,b])$.

Ex. 5.1.25. Let $f, g\colon [0,1] \to \mathbb{R}$ be continuous functions. Assume that $f(x) \in [0,1]$ for all x and $g(0) = 0$ and $g(1) = 1$. Show that $f(x) = g(x)$ for some $x \in [0,1]$.

Example 5.1.26. We show that $S^n := \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ is connected. The strategy is to show that S^n is the union of the closed upper and lower hemispheres, each of which is homeomorphic to the closed unit disk in $\mathbb{R}^n$ and to observe that the hemispheres intersect.

Let $S^n_+ := \{x \in \mathbb{R}^{n+1} : x_{n+1} \geq 0\}$ be the upper hemisphere. Let $D_n := \{u \in \mathbb{R}^n : \|u\| \leq 1\}$. Note that D_n is convex and hence connected. We claim that D_n is homeomorphic to S^n_+. Consider the map $f_+ \colon D_n \to S^n_+$ given by $f_+(u) = (u_1, \ldots, u_n, \sqrt{1 - \|u\|^2})$. Clearly, f_+ is bijective and continuous. Since f_+ is a bijective continuous map of a compact space to a metric space, it is a homeomorphism. In any case, S^n_+ being the continuous image of the connected set D_n, is connected. Similarly, we show that the lower hemisphere $S^n_- := \{x \in \mathbb{R}^{n+1} : x_{n+1} \leq 0\}$ is the image of $f_- \colon D_n \to S^n_-$ given by $f_-(u) = (u_1, \ldots, u_n, -\sqrt{1 - \|u\|^2})$. Clearly, the intersection $S^n_+ \cap S^n_- = \{x \in \mathbb{R}^{n+1} : x_{n+1} = 0\}$ is nonempty. Hence we conclude that S^n is connected by (1) of Theorem 5.1.8.

We have already seen that S^n is the union of two sets homeomorphic to $\mathbb{R}^n$ (see Example 3.6) with nonempty intersection. The connectedness of S^n follows from this.

Ex. 5.1.27. Let A be the union of the following subsets of $\mathbb{R}^2$:

$$\begin{aligned} S &:= \{(x,y) : x^2+y^2=1\} \\ L_1 &:= \{(x,y) : x \geq 1 \text{ and } y=0\} \\ L_2 &:= \{(x,y) : x \leq -1 \text{ and } y=0\} \\ L_3 &:= \{(x,y) : y \geq 1 \text{ and } x=0\} \\ L_4 &:= \{(x,y) : x \leq -1 \text{ and } x=0\}. \end{aligned}$$

Show that A is connected subset of $\mathbb{R}^2$. (Draw a picture of A!)

Can you generalize this exercise?

Ex. 5.1.28. Let X be a (metric) space. Let S and L_i $(i \in I)$ be connected subsets of X. Assume that $S \cap L_i \neq \emptyset$. Show that $S \cup (\cup_{i \in I} L_i)$ is a connected subset of X. (This is a generalization of the last exercise!)

Ex. 5.1.29. Show that $\mathbb{R}^2 \setminus \{0\}$ is connected. (See also Example 5.2.15. The next example proves something stronger.)

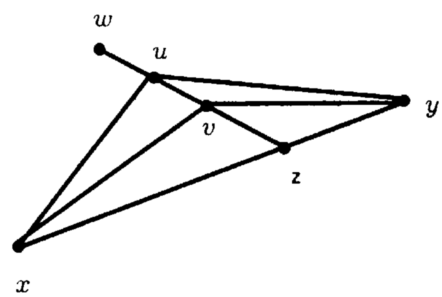

Figure 5.1: Illustration for Example 5.1.30

Example 5.1.30. Let $n > 1$. Let A be a countable subset of $\mathbb{R}^n$. We claim that $\mathbb{R}^n \setminus A$ is connected.

By the last exercise, it suffices to show that any two points of $\mathbb{R}^n \setminus A$ lie in a connected set. Recall that

$$[p,q] := \{x \in \mathbb{R}^n : x = (1-t)p + tq, 0 \leq t \leq 1\} \text{ for } p,q \in \mathbb{R}^n.$$

Given $x, y \in \mathbb{R}^n \setminus A$, choose a point $z \in [x,y]$ other than x and y. (Why is this possible?) Choose $w \in \mathbb{R}^n \setminus [x,y]$ such that $[z,w] \cap [x,y] = \{z\}$. (Why is this possible? Use $n > 1$. See Figure 5.1.) For each $u \in [z,w]$, let $B_u := [x, u] \cup [u, y]$. Then B_u is a connected set (why?) such that $x, u \in B_u$.

We claim that we can find $u \in [z, w]$ such that the set B_u lies in the complement of A. Assume the contrary. Then for each $u \in [z, w]$, the set $B_u \cap A \neq \emptyset$. Note that $B_u \cap B_v = \{x, y\}$, for $u, v \in [z, w]$ and $u \neq v$. (Why? Draw pictures.) Since $\{x, y\} \cap A = \emptyset$, it follows that there exist no point common to $B_u \cap E$ and $B_v \cap A$ for u, v as above. We then conclude that the set $A \cap (\cup_{u \in [z,w]} B_u)$ is uncountable. (How?) This is a subset of the countable set A, a contradiction. □

Theorem 5.1.31. *Let X and Y be connected (metric) spaces. Then the product space $X \times Y$ is connected.*

Proof. Let $f : X \times Y \to \{\pm 1\}$ be a continuous map. Let $(x_0, y_0) \in X \times Y$ be fixed. Let (x, y) be an arbitrary point in $X \times Y$. If we show that $f(x, y) = f(x_0, y_0)$, we are through.

To prove the above claim, let us first observe that for every point $y \in Y$, the map $i_y : X \to X \times Y$ defined by $i_y(x) := (x, y)$ is continuous. (See Ex. 3.1.8.) Similarly the map $i_x : Y \to X \times Y$ defined by $i_x(y) := (x, y)$ is continuous for every point x in X. Therefore for every point y in Y, the subset $X \times \{y\} := \{(x, y) : x \in X\}$ is a connected subset of $X \times Y$; similarly, the subset $\{x\} \times Y := \{(x, y) : y \in Y\}$ is a connected subset of $X \times Y$ for every point x in X.

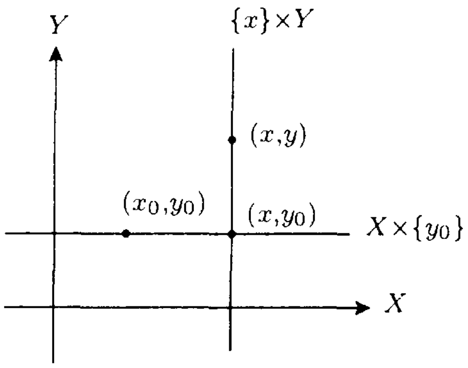

Figure 5.2: Connectedness of the product

Now the point (x, y_0) lies in both sets $X \times \{y_0\}$ and $\{x\} \times Y$. The restrictions of f to either of these sets are continuous and hence constants. We see that $f(x_0, y_0) = f(x, y_0))$ for all $x \in X$ and similarly, $f(x, y) = f(x, y_0)$ for all $y \in Y$. In particular, $f(x, y) = f(x, y_0) = f(x_0, y_0)$. (See Figure 5.2 on page 113.) □

Ex. 5.1.32. Show that the annulus $\{x \in \mathbb{R}^2 : 1 < \|x\| < 2\}$ is connected. *Hint:* Continuous image of a connected set is connected.

Ex. 5.1.33. Show that S^n, the unit sphere in $\mathbb{R}^{n+1}$ is connected. *Hint:* Can you think of a continuous map from $\mathbb{R}^{n+1} \setminus \{0\}$ to onto S^n?

Ex. 5.1.34. We say that $f\colon X \to Y$ is *a locally constant* function if for each $x \in X$, there exists an open set U_x containing x with the property that f is a constant on U_x.

If X is connected, then any locally constant function is constant on X.

Ex. 5.1.35. Let U be an open connected subset of $\mathbb{R}^n$ and $f\colon U \to \mathbb{R}$ be a differentiable function such that $Df(p) = 0$ for all $p \in U$. Then f is a constant function.

Ex. 5.1.36. Let $f\colon X \to \mathbb{R}$ be a nonconstant continuous function on a connected (metric) space. Show that $f(X)$ is uncountable and hence X is uncountable.

Ex. 5.1.37. Let (X, d) be a connected metric space. Assume that X has at least two elements. Then $|X| \geq |\mathbb{R}|$.

Ex. 5.1.38. Let (X, d) be an unbounded connected metric space. Let $x \in X$ and $r > 0$ be arbitrary. Show that there exists $y \in X$ such that $d(x, y) = r$.

Ex. 5.1.39. Which of the following sets are connected subsets of $\mathbb{R}^2$?
(a) $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.
(b) $\{(x, y) \in \mathbb{R}^2 : y = x^2\}$.
(c) $\{(x, y) \in \mathbb{R}^2 : xy = 1\}$.
(d) $\{(x, y) \in \mathbb{R}^2 : xy = c \text{ for some fixed } c \in \mathbb{R}\}$.
(e) $\{(x, y) \in \mathbb{R}^2 : (x^2/a^2) + (y^2/b^2) = 1\}$ for some $a > b > 0$.

Theorem 5.1.10 shows that connectedness is a topological property. We can use this to show that certain spaces are not homeomorphic.

Ex. 5.1.40. Show that a circle or a line or a parabola in $\mathbb{R}^2$ is not homeomorphic to a hyperbola.

Ex. 5.1.41. Show that $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^2$. *Hint:* Observe that if $f\colon X \to Y$ is a homeomorhism and if $f(A) = B$ for a subset $A \subset X$, the the restriction of f to $X \setminus A$ is a homeomorphsim of $X \setminus A$ to $Y \setminus B$. Recall that Ex. 5.1.29 says that $\mathbb{R}^2$ minus a point is connected.

Ex. 5.1.42. Let X be the union of axes given by $xy = 0$ in $\mathbb{R}^2$. Is it homeomorphic to a line, a circle, a parabola or the rectangular hyperbola $xy = 1$?

Ex. 5.1.43. Let $A \subset X$. What does it mean to say that the characteristic function χ_A continuous?

Ex. 5.1.44. Give an example of a sequence (A_n) of connected subsets of $\mathbb{R}^2$ such that $A_{n+1} \subset A_n$ for $n \in \mathbb{N}$, but $\cap_n A_n$ is not connected.

Ex. 5.1.45. Show that no nonempty open subset of $\mathbb{R}$ is homeomorphic to an open subset of $\mathbb{R}^2$.

Ex. 5.1.46. The notation is as in Ex. 2.1.19. Show that the maximal connected subsets of $(\mathbb{Q}, d_p)$ are singletons.

What are the maximal connected subsets of $\mathbb{Q}$ with the standard metric?

5.2 Path Connected spaces

Definition 5.2.1. Let X be a (metric) space. A *path* in X is a continuous map $\gamma \colon [0,1] \to X$. in X. If $\gamma(0) = x$ and $\gamma(1) = y$, then γ is said to be a path joining the points x and y or simply a path from x to y. See Figure 5.3. We also say that x is path connected to y.

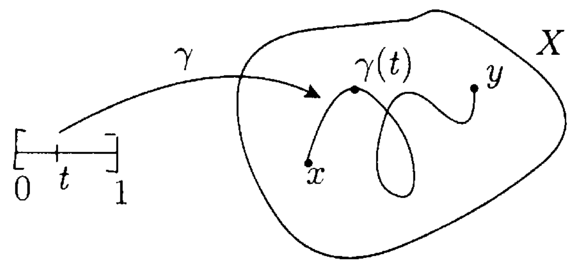

Figure 5.3: Path between x and y

Ex. 5.2.2. If x is path connected to y, then y is path connected to x. *Hint:* Define $\sigma(t) := \gamma(1-t)$ for $t \in [0,1]$. Then show that σ connects y to x. (The path σ is called the reverse path of γ.)

Ex. 5.2.3. Give at least two paths in $\mathbb{R}^2$ that connect $(-1,0)$ and $(1,0)$ and pass through (0,1).

Ex. 5.2.4. Show that if x is path-connected to y and y is path connected to z in X, then x is path-connected to z.

More precisely, prove the following. Let $\gamma_i \colon [0,1] \to X$, $i = 1, 2$, be two paths such that $\gamma_1(1) = \gamma_2(0)$. Then show that there exists a path

$\gamma_3 \colon [0,1] \to X$ such that $\gamma_3(0) = \gamma_1(0)$, $\gamma_3(1/2) = \gamma_1(1) = \gamma_2(0)$ and $\gamma_3(1) = \gamma_2(1)$. *Hint:* If you think of $[0,1]$ as the time interval, your train γ_1 should cover the distance between $\gamma_1(0)$ and $\gamma_1(1)$ along the railway track $\gamma_1[0,1]$ within half the time and γ_2 should take over from there. To wit, consider

$$\gamma_3(t) := \begin{cases} \gamma_1(2t) & \text{if } t \in [0,1/2] \\ \gamma_2(2t-1) & \text{if } t \in [1/2,1]. \end{cases}$$

Definition 5.2.5. A (metric) space X is said to be *path connected* if for any pair of points x and y in X, there exists a path $\gamma \colon [0,1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

Ex. 5.2.6. Show that any interval in $\mathbb{R}$ is path connected.

Example 5.2.7. The space $\mathbb{R}^n$ is path connected. Any two points can be joined by a line segment: $\gamma(t) := x + t(y-x)$, for $0 \le t \le 1$. We call this path γ a linear path.

Ex. 5.2.8. Show that any convex set in an NLS is path-connected. Hence conclude that any open or closed ball in an NLS is connected.

Ex. 5.2.9. For every $r > 0$, show that the circle $C_r := \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = r^2\}$ is path connected.

Ex. 5.2.10. Show that the set $\{(x,y) \in \mathbb{R}^2 : x \ge 0 \text{ and } x^2 - y^2 = 1\}$ is path connected.

Show that the hyperbola $\{(x,y) \in \mathbb{R}^2 : x^2 - y^2 = 1\}$ is not path connected.

Ex. 5.2.11. Show that the parabola $\{(x,y) \in \mathbb{R}^2 : y^2 = x\}$ is path connected.

Ex. 5.2.12. Show that the union of the two parabolas $\{(x,y) \in \mathbb{R}^2 : y^2 = x\}$ and $\{(x,y) \in \mathbb{R}^2 : y = x^2\}$ is path connected.

Ex. 5.2.13. The union of the parabolas $\{(x,y) \in \mathbb{R}^2 : y^2 = x\}$ and $\{(x,y) \in \mathbb{R}^2 : y^2 = -x\}$ is path connected.

Lemma 5.2.14. *A (metric) space is path connected iff there exists a point $a \in X$ which is path-connected to any $x \in X$.*

Proof. Let $x, y \in X$ be arbitrary. Let α (respectively, β) be path joining a to x (respectively to y). Let σ be the reverse path joining x to a. (See Ex. 5.2.2.) Then there exists a path joining x to y by Ex. 5.2.4. Thus any pair of points is path connected and hence X is path connected. $\square$

Example 5.2.15. We show that $\mathbb{R}^n \setminus \{0\}$ is path connected if $n \geq 2$. The strategy is to modify the approach of Example 5.1.30. Let $p = e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^n$ and $q = e_2 = (0, 1, 0, \ldots, 0) \in \mathbb{R}^n$. Let $x \in \mathbb{R}^n$ be any nonzero vector. Consider the line segments $[p, x]$ and $[x, q]$. We claim that at least one of them does not pass through the origin. If false, then $(1-t)p + tx = 0 = (1-s)q + sx$ for some $0 \leq s, t \leq 1$. From these equation, it follows that $(1-t)p = -tx$ and $(1-s)q = -sx$. Thus p and q are scalar multiples of the same vector and hence they are linearly dependent. This contradiction shows that our claim is true. Note also that by a similar reasoning, the line segment $[p, q]$ does not pass through the origin. Now consider the 'path' $[x, p]$ or the path $[x, q] \cup [q, p]$ connecting x and p, not passing through the origin. Thus any nonzero $x \in \mathbb{R}^n$ is path-connected to p and hence $\mathbb{R}^n \setminus \{0\}$ is path-connected by Lemma 5.2.14.

Ex. 5.2.16. Show that any continuous image of a path-connected space is path connected, that is, if $f \colon X \to Y$ is a continuous and X is path-connected, then $f(X)$ is path-connected.

In particular, conclude that path-connectedness a topological property. Can you think of applications?

Ex. 5.2.17. Let A and B be path-connected subsets of a (metric) space with $A \cap B \neq \emptyset$. Show that $A \cup B$ is path-connected.

Example 5.2.18. The unit sphere $S^n := \{x \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_{n+1}^2 = 1\}$ is path connected.

The reader should visualize the case when $n = 2$. The strategy is to show that any two distinct points of $S^2 \subset \mathbb{R}^3$ lie in a great circle, that is, the intersection of S^2 and a plane through the origin in $\mathbb{R}^3$. We carry out the details in the general case.

A 'plane' through the origin in $\mathbb{R}^{n+1}$ is a two dimensional vector subspace of $\mathbb{R}^{n+1}$. A 'great circle' in the sphere is the intersection of S^n and a plane P passing through the origin. Let u, v be an orthonormal basis of the plane P. We claim that $P \cap S^n$ can be described as follows:

$$P \cap S^n = \{x \in S^n : x = \cos tu + \sin tv, \text{ for some } t \in [0, 2\pi].\}$$

Clearly, if $x = \cos t\, u + \sin t\, v$, then $x \in P$ and

$$x \cdot x = \cos^2 t(u \cdot u) + 2 \cos t \sin t\, (u \cdot v) + \sin^2 t(v \cdot v) = 1,$$

since $u \cdot u = 1 = v \cdot v$ and $u \cdot v = 0$. Conversely, if $x \in P$, then we can write $x = au + bv$ for some $a, b \in \mathbb{R}$. If we further assume that $x \in S^n$, then $\|x\|^2 = 1$ so that

$$x \cdot x = a^2 + b^2 = 1.$$

Thus $(a, b) \in \mathbb{R}^2$ with $a^2 + b^2 = 1$. Therefore we can find $t \in [0, 2\pi)$ such that $a = \cos t$ and $b = \sin t$. This proves our claim.

In particular, $P \cap S^n$ is the image of $[0, 2\pi]$ under the continuous map

$$\gamma_P(t) := \cos tu + \sin tv, \qquad (0 \leq t \leq 2\pi).$$

Note that γ_P is a path that connects any two points on it.

Given any two points $x, y, x \neq y$ of S^n, we show that it lies on a great circle. If $y \neq -x$, then x and y are linearly independent (why?) and hence x and y span a two dimensional vector subspace P of $\mathbb{R}^{n+1}$. See Figure 5.4. Clearly, $x, y \in P \cap S^n$ and hence are connected by the path γ_P. (Note that γ_P need not be of the form $\gamma_P = \cos t\, x + \sin t\, y$, unless $x \cdot y = 0$. Can you find an explicit expression for γ_P that involves only x and y? You have to recall Gram-Schmidt process!)

Figure 5.4: Great circle on sphere

If $y = -x$, since $n \geq 1$, we can find a vector v which is of unit norm and such that $x \cdot v = 0$. Then x, v form an orthonormal basis of the vector subspace P spanned by x, v. If we let $\gamma_P(t) := \cos tx + \sin tv$, then $\gamma_P(0) = x$ and $\gamma_P(\pi) = y$. Thus x and y are path connected via γ_P.

Of course, an easier proof would run as follows: As seen in Example 5.1.26, S^n is the union of two hemisphers and their intersection is nonempty. Now the hemi-spheres are homeomorphic the convex set $B[0, 1]$ and hence are path-connected (by Ex. 5.2.8 and Ex. 5.2.16). Path-connectedness of S^n follows from Ex. 5.2.17.

Ex. 5.2.19. Give a third proof of path-connectedness of S^n as follows: The sphere S^n is the continuous image of the path-connected space $\mathbb{R}^{n+1} \setminus \{0\}$.

Ex. 5.2.20. Show that the annulus $\{x \in \mathbb{R}^2 : 1 \leq \|x\| \leq 2\}$ is path connected. How about $\{x \in \mathbb{R}^2 : 1 < \|x\| < 2\}$?

Proposition 5.2.21. *Let a (metric) space X be path connected. Then X is connected.*

Proof. Since this is a very useful result, we shall give a proof. However, it is an easy exercise and the reader should do it on his own.

Let $f : X \to \{\pm\}$ be continuous. Fix $a \in X$. Let $x \in X$ be arbitrary. Since X is path-connected, there exists a path $\gamma : [0, 1] \to X$ such that $\gamma(0) = a$ and $\gamma(1) = x$. The function $f \circ \gamma : [0, 1] \to \{\pm\}$ is continuous on the connected set $[0, 1]$ by Proposition 5.1.7 and hence must be a constant. In particular, $f(a) = f \circ \gamma(0) = f \circ \gamma(1) = f(x)$. Since $x \in X$ is arbitrary, we have shown that f is a constant function. Therefore, X is connected. $\square$

Remark 5.2.22. The converse of the last exercise is not true. We give two examples, each is a slight modification of the other.

Example 5.2.23 (Topologist's Sine Curve-I). Let

$$A := \{(x, \sin(\pi/x)) : 0 < x \leq 1\} \text{ and } B := \{(0, y) : -1 \leq y \leq 1\}.$$

Let $X = A \cup B \subset \mathbb{R}^2$ be given the induced metric topology. We claim that X is connected but not path connected.

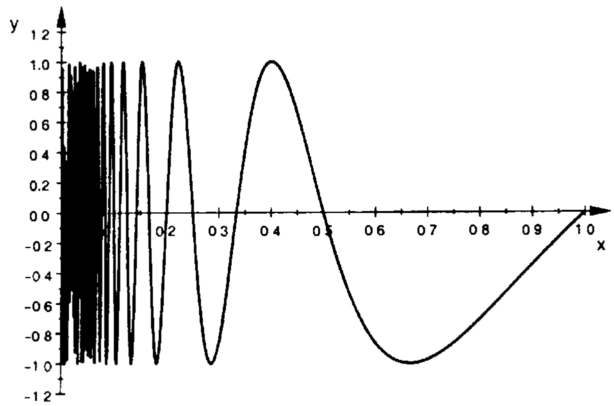

Figure 5.5: Topologist's Sine Curve-I

Let $\gamma\colon [0,1] \to X$ be a path joining $(0,0)$ to $(1,0)$. We write $\gamma(t) = (\gamma_1(t), \gamma_2(t))$. Since B is closed in X, the inverse image $\gamma^{-1}(B)$ is closed, $0 \in \gamma^{-1}(B)$. Let t_0 be the least upper bound of this closed and bounded set. Obviously, $t_0 \in \gamma^{-1}(B)$. Note that $0 < t_0 < 1$. We claim that γ_2 is not continuous at t_0.

For any $\delta > 0$, with $t_0 + \delta \leq 1$, we must have $\gamma_1(t_0 + \delta) > 0$. (Why?) Hence there exists $n \in \mathbb{N}$ such that $\gamma_1(t_0) < \frac{2}{4n+1} < \gamma_1(t_0 + \delta)$. By the intermediate value theorem applied to the continuous function γ_1, we can find t such that $t_0 < t < t_0 + \delta$ and such that $\gamma_1(t) = \frac{2}{4n+1}$. Hence $\gamma_2(t) = 1$ and $|\gamma_2(t) - \gamma_2(t_0)| \geq 1$. We therefore conclude that γ_2 is not continuous at t_0.

Example 5.2.24 (Topologist's Sine Curve-II). Consider

$$X := \{(x, \sin(1/x)) : x > 0\} \cup \{(x, 0) : -1 \leq x \leq 0\} = A \cup B \text{ (say.)}$$

Clearly each of A and B is connected. Also, the point $(0,0)$ is a limit point of the set A and hence $A_1 = A \cup \{(0,0)\} \subset \overline{A}$ is connected. Since B and A_1 have a point in common their union X is connected.

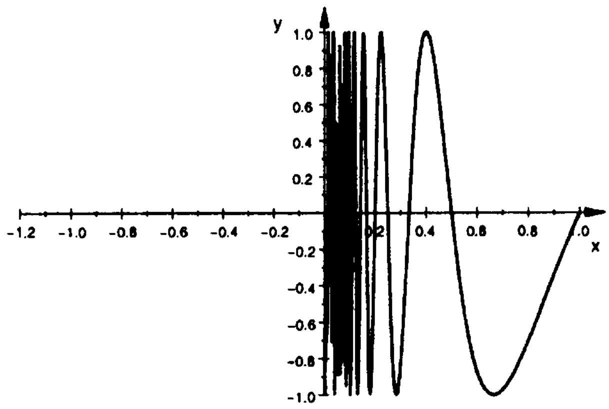

Figure 5.6: Topologist's Sine Curve-II

We claim that X is not path-connected. In fact, we show that there is no path connecting $(1/\pi, 0)$ with $(0,0)$. Let $\gamma\colon [0,1] \to X$ be path such that $\gamma(0) = (1/\pi, 0)$ and $\gamma(1) = (0,0)$. Then $\pi_1 \circ \gamma$ must take all values that lie between 0 and $1/\pi$. In particular, there exist $t_n \in [0,1]$ such that $\pi_1 \circ \gamma(t_n) = \frac{1}{(2n+\frac{1}{2})\pi}$. Then, $\gamma(t_n) \to (0,1)$ as $n \to \infty$. By Bolzano-Weierstrass, there exists a convergent subsequence, (t_{n_k}). Let t_0 be the limit of this subsequence. Then $\pi_1 \circ \gamma(t_{n_k}) \to 0$. Thus, $\gamma(t_0)$ must be $(0, y)$ for some y. Since $\gamma(t_0) = (0, y) \in X$, it follows that $y = 0$. But, $\pi_2 \circ \gamma(t_0) = \lim \pi_2 \circ \gamma(t_{n_k}) = 1$. This contradiction shows that there is no such path γ. □

Ex. 5.2.25. Assume that a path $\gamma\colon [0,1] \to \mathbb{R}^n$ connects a point $x \in B(0,1) \subset \mathbb{R}^n$ to a point y with $\|y\| > 1$. Show that there exists $t \in [0,1]$ such that $\|\gamma(t)\| = 1$. (See Figure 5.7.)

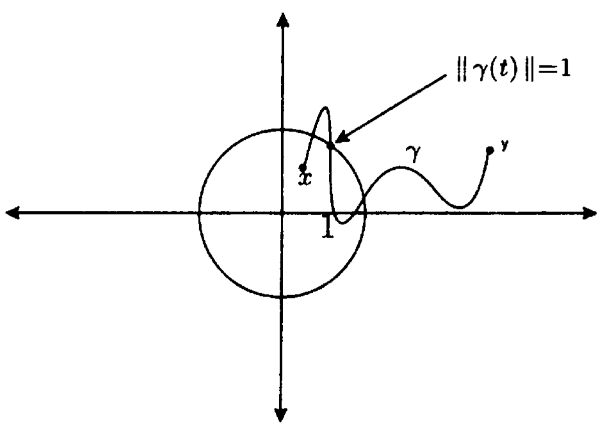

Figure 5.7: Illustration for Exercise 5.2.25

The following result is a typical application of connectedness argument and also provides a large class of path-connected spaces.

Proposition 5.2.26. *Let U be an open connected subset of $\mathbb{R}^n$. Then U is path connected.*

Proof. Assume that $U \neq \emptyset$ and let $a \in U$. Consider the set A consisting of those points x of U which can be path-connected to a in U, that is, there exists a path in U joining x and a. Since a is path connected to itself by a constant path $\gamma(t) = a$ for $t \in [0, 1]$, we see that $a \in A$. We plan to show that it is both open and closed in U and hence $A = U$.

We show that A is closed in U. Let $x \in U$ be a limit point of A. Then there exists a sequence (x_n) in A such that $x_n \to x$. Since $x \in U$ and U is open, there exists $r > 0$ such that $B(x, r) \subset U$. Since $x_n \to x$, for the r as above, there exists $N \in \mathbb{N}$ such that $x_n \in B(x, r)$ for $n \geq N$. Now $x_N \in A$ and hence x_N is path connected to a. Also, since $B(x, r)$ is path connected (being a convex set), there exists a path joining x_N to x. We therefore conclude that there exists a path joining a to x. In other words, $x \in A$. Therefore A is closed.

We show that A is open in U. Let $x \in A$. Since $x \in U$ and U is open, there exists $r > 0$ such that $B(x, r) \subset U$. Since a is connected to x and x is connected to any point $y \in B(x, r)$ (thanks to the convexity of $B(x, r)$), it follows that a is connected to any $y \in B(x, r)$. That is to say that $B(x, r) \subset A$. Hence we conclude that A is open.

Since U is connected and A is a nonempty set which is both open and closed, we conclude that $A = U$. That U is path-connected follows form Lemma 5.2.14. $\square$

Ex. 5.2.27. By going through the argument above, we realize that we could prove the result in a more general setting. Let X be a connected (metric) space. Assume that each point of X has an open set U such that $x \in U$ and U is path-connected. Then X is path connected.

Ex. 5.2.28. Let A be a connected subset in $\mathbb{R}^n$ and $\varepsilon > 0$. Show that the ε-neighbourhood of A defined by $U_\varepsilon(A) := \{x \in \mathbb{R}^n : d_A(x) < \varepsilon\}$ is path-connected.

Chapter 6

Complete Metric Spaces

We recall the definition of a complete metric space and give two most important examples of complete metric spaces of functions and one example of an incomplete metric space of functions.

6.1 Examples of Complete Metric Spaces

Recall that a metric space (X, d) is said to be complete if every Cauchy sequence in X is convergent in X. W have shown that $\mathbb{R}^n$ is complete for $n \in \mathbb{N}$.

We start with an example next only in importance to $\mathbb{R}$.

Proposition 6.1.1. *The normed linear space* $(B(X), \| \ \|_\infty)$ *is complete.*

Proof. Let a Cauchy sequence (f_n) be given. By Example 2.3.4, the sequence of functions (f_n) is uniformly Cauchy on X.

Fix $x \in X$. Consider the sequence of scalars $(f_n(x))$. By (2.2), it is a Cauchy sequence in $\mathbb{R}$ (or in $\mathbb{C}$, if we are dealing with complex valued functions). For definiteness sake, we shall assume that we are working with real valued functions. (The proof for complex valued functions is exactly the same.)

Since $\mathbb{R}$ is complete, the sequence converges to a real number $r \in \mathbb{R}$, which we denote by $f(x)$, to show the dependence of $\alpha = \lim_{n\to\infty} f_n(x)$ on x. Thus, for each $x \in X$, we get a real number $f(x)$ such that $\lim_{n\to\infty} f_n(x) = f(x)$. (What we have shown so far is that the sequence f_n converges to some function $f\colon X \to \mathbb{R}$ pointwise.) To show that $f_n \to f$ in the metric space, we first of all need to show that the function $x \mapsto f(x)$ lies in $B(X)$, that is, f is bounded on X. Next we need to show that $f_n \to f$ in the metric space, that is, in view of Example 2.1.11, we need to prove that $f_n \to f$ uniformly on X. If we accomplish these two tasks, the proof of the theorem is complete.

We first show that f is bounded. Since (f_n) is Cauchy in the metric space, it is bounded by Ex. 2.3.5. Therefore, there exists $M > 0$ such that $\|f_n\| \leq M$ for $n \in \mathbb{N}$. In particular, $|f_n(x)| \leq M$ for all $x \in X$ and $n \in \mathbb{N}$. Fix $x \in X$. Since $f_n(x) \to f(x)$, given $\varepsilon = 1$, there exists N (which may depend on x and so we may denote it by $N(x)$) such that $|f_n(x) - f(x)| < 1$ for $n \geq N(x)$. It follows that

$$\begin{aligned} |f(x)| &= \left|[f(x) - f_{N(x)}(x)] + f_{N(x)}(x)\right| \\ &\leq \left|f(x) - f_{N(x)}(x)\right| + \left|f_{N(x)}(x)\right| \\ &< 1 + M. \end{aligned}$$

We have therefore shown that $|f(x)| \leq M + 1$ for all $x \in X$ and hence $\|f\| \leq M + 1$. We conclude that f is bounded and hence is an element of $B(X)$. (Go through the argument carefully, as we shall again use it.)

We now show that $f_n \to f$ uniformly on X. (This proof is delicate and see the remark after the proof.) Let $\varepsilon > 0$ be given. Since (f_n) is Cauchy in the metric, there exists $n_0 \in \mathbb{N}$ such that $\|f_n - f_m\|_\infty < \varepsilon/2$ for $m, n \geq n_0$. We claim that $|f(x) - f_n(x)| < \varepsilon$ for $n \geq n_0$. Fix $x \in X$. Since $f_n(x) \to f(x)$, for the given $\varepsilon > 0$, there exists $N(x) \in \mathbb{N}$ such that $|f_m(x) - f(x)| < \varepsilon/2$ for $m \geq N(x)$. We have, for all $n \geq n_0$,

$$\begin{aligned} |f(x) - f_n(x)| &= |f(x) - f_m(x) + f_m(x) - f_n(x)| \\ &\qquad \text{(for any } m \in \mathbb{N}) \\ &= |f(x) - f_m(x) + f_m(x) - f_n(x)| \\ &\qquad \text{(in particular for any } m \in \mathbb{N} \text{ with } m \geq N(x)) \\ &= |f(x) - f_m(x) + f_m(x) - f_n(x)| \\ &\qquad \text{(for any } m \in \mathbb{N} \text{ with } m \geq N(x) \text{ and } m \geq n_0) \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

That is, $f_n \to f$ uniformly on x. □

Remark 6.1.2. The proof in the last paragraph is not the one found in any textbook. The standard proof for the boundedness of the limit function f and the convergence of f_n to f in the metric is by passing to the limit as $n \to \infty$ keeping m fixed in the inequalities of the kind:

$$|f_n(x) - f_m(x)| < \varepsilon \text{ for } m, n \geq N.$$

While this proof is correct, the beginner needs to be told that we are using the continuity of the distance function (in this case, the absolute value). We refer to the trick we have employed above as 'curry leaves trick'. In south Indian cooking, one uses curry leaves to add flavour and then throw the flavouring agent. In our proof, the curry leaves are m's.

Note also that we have employed this trick even earlier. See the proof of (3) of Proposition 2.3.6.

Theorem 6.1.3 (Weierstrass M-Test). *Let X be any nonempty set. Let (f_n) be a sequence in $B(X,\mathbb{R})$ such that there exists $M_n > 0$ such that $\sum_{n=1}^{\infty} M_n < \infty$. Then the series $\sum_{n=1}^{\infty} f_n$ is uniformly convergent, that is, the sequence (s_n) of partial sums of the series is uniformly convergent on X. (Recall that $s_n(x) := \sum_{k=1}^{n} f_k(x)$.)*

Proof. We need only show that the sequence (s_n) is Cauchy in the the metric space $(B(X,\mathbb{R}), \|\ \|_\infty)$. Let $\varepsilon > 0$ be given. Since $\sum_{n=1}^{\infty} M_n < \infty$, the partial sums of the series $\sum_{n=1}^{\infty} M_n$ is convergent and in particular Cauchy in $\mathbb{R}$. For the given ε, there exists $N \in \mathbb{N}$ such that $\sum_{k=m+1}^{n} M_k < \varepsilon$ for $n \geq m \geq N$. Hence we have, for $n \geq m \geq N$,

$$\begin{aligned}
\| s_n - s_m \|_\infty &= \sup_{x \in X} \left| \sum_{k=m+1}^{n} f_k(x) \right| \\
&\leq \sup_{x \in X} \sum_{k=m+1}^{n} |f_k(x)| \\
&\leq \sum_{k=m+1}^{n} M_k \\
&< \varepsilon.
\end{aligned}$$

Thus (s_n) is Cauchy in $B(X)$. Since $(B(X), \|\ \|_\infty)$ is complete, the result follows. □

Ex. 6.1.4. Let (X,d) be a complete metric space and $E \subset X$. Then E is closed in X iff (E,d) is a complete metric space.

Ex. 6.1.5. Show that the set $\mathbf{c}$ of convergent sequences in the NLS of all bounded real sequences under the sup norm $\|\ \|_\infty$ is complete. *Hint:* Enough to show that $\mathbf{c}$ is closed. If $x = (x_n)$ is a limit point of $\mathbf{c}$, it suffices to show that x is Cauchy.

Ex. 6.1.6. This is a remark on Ex. 2.3.14 on page 47.

By Ex. 6.1.4, the set $[1,\infty)$ is closed in $\mathbb{R}$.

Do you understand why the conclusion of Ex. 2.3.14 holds?

Theorem 6.1.7. *Let X be a compact (metric) space. The normed linear space $(C(X), \|\ \|_\infty)$ of real/complex valued continuous functions on X is complete.*

Proof. First of all observe that since X is compact, $C(X) \subset B(X)$, the set of bounded (real/complex) functions on X. We have already seen that $(B(X), \| \|_\infty)$ is complete. So it suffices to show that $C(X)$ is closed in $B(X)$ under the sup norm topology. This is an easy exercise and the reader should do on his own.

What needs to be shown is that if (f_n) is a sequence in $C(X)$ converging to an $f \in B(X)$ with respect to the sup norm, then $f \in C(X)$. Since convergence f_n to f in $\| \|_\infty$ is the same as the uniform convergence of f_n to f on X, the continuity of f is a standard result from real analysis and which uses a trick fondly remembered as 3ε-trick. In case, you wish to recall the proof, it goes as follows.

Let $x_0 \in X$. We want to prove the continuity of f at x_0. Let $\varepsilon > 0$ be given. By the uniform convergence, there exists $N \in \mathbb{N}$ such that $|f(x) - f_n(x)| < \varepsilon/3$ for all $x \in X$. In particular this inequality is true for $n = N$. By the continuity of f_N at x_0, there exists $\delta > 0$ (or an open set $U \ni x_0$ in X) such that if $x \in B(x_0, \delta)$ (or if $x \in U$), then, $|f_N(x) - f_N(x_0)| < \varepsilon/3$. Thus, we have for any x as above,

$$\begin{aligned}
|f(x) - f(x_0)| &\leq |[f(x) - f_N(x)] + [f_N(x) - f_N(x_0)] \\
&\quad + [f_N(x_0) - f(x_0)]| \\
&\leq |f(x) - f_N(x)| + |f_N(x) - f_N(x_0)| \\
&\quad + |f_N(x_0) - f(x_0)| \\
&< \varepsilon/3 + \varepsilon/3 + \varepsilon/3.
\end{aligned}$$

Note that in the last inequality, each of the first and the third terms is less than $\varepsilon/3$ by the uniform convergence while the second term is less than $\varepsilon/3$ thanks to the continuity of f_N at x_0. □

Ex. 6.1.8. Recall Ex. 3.1.22 on page 58. Show that Ex. 6.1.5 is a corollary of the last result if we observe that $\mathbf{c} = C(X)$ where X is the compact space $X = \{1/n : n \in \mathbb{N}\} \cup \{0\}$.

Ex. 6.1.9. Let $f\colon X \to Y$ be a continuous onto map of metric spaces. Assume that $d(x_1, x_2) \leq d(f(x_1), f(x_2))$ for all $x_1, x_2 \in X$. Show that if X is complete, so is Y. *Hint:* Observe that f is one-one!

The following result gives characterizations of complte metric spaces.

Theorem 6.1.10. *Let (X, d) be a metric space. The following are equivalent.*

(a) *(X, d) is complete.*

(b) *Every sequence (x_n) in X with $\sum_{n=1}^\infty d(x_{n+1}, x_n) < \infty$, is convergent.*

(c) *Every Cauchy sequence in X has a convergent subsequence.*

Proof. **(a) $\implies$ (b).** Consider a sequence (x_n) with $\sum_{n=1}^{\infty} d(x_{n+1}, x_n) < \infty$. We need to show that (x_n) is convergent. Since X is complete, it suffices to show that (x_n) is Cauchy. Note that $\sum_{k=m}^{\infty} d(x_k, x_{k+1}) \to 0$ since $\sum_{n=1}^{\infty} d(x_{n+1}, x_n)$ is convergent. Hence given $\varepsilon > 0$, there exists N such that $\sum_{n=N}^{\infty} d(x_{n+1}, x_n) < \varepsilon$.[1] For positive integers $n > m \geq N$, we have

$$d(x_m, x_n) \leq \sum_{k=m}^{n-1} d(x_k, x_{k+1}) \sum_{k=N}^{\infty} d(x_k, x_{k+1}) < \varepsilon.$$

Hence (x_n) is Cauchy.

(b) $\implies$ (c). Let (x_n) be Cauchy. Given $\varepsilon := 2^{-k}$, there exists n_k such that $d(x_m, x_n) < 2^{-k}$ for $m, n \geq n_k$. We choose n_k's with a little more care. Suppose we have chosen n_k. Then, for $\varepsilon = 2^{-k-1}$, we choose $n_{k+1} > n_k$ such that $d(x_m x_n) < 2^{-k-1}$. We claim that (x_{n_k}) is Cauchy. For, if $k < l$,

$$d(x_{n_k}, d_{n_l}) \leq \sum_{j=k}^{l-1} d(x_{n_j}, x_{n_{j+1}}) \leq \sum_{j=k}^{\infty} 2^{-j} \to 0,$$

as $j \to \infty$ as seen earlier.

(c) $\implies$ (a). This is seen earlier. See (c) of Proposition 2.3.6. $\square$

Ex. 6.1.11 (Abstract Weierstrass M-Test). Let $(X, \| \ \|)$ be an NLS. Show that X is complete iff for every sequence (x_n) in X such that $\sum_{n=1}^{\infty} \|x_n\| < \infty$, the series $\sum_{n=1}^{\infty} x_n$ is convergent in X. (The series $\sum_{n=1}^{\infty} x_n$ is convergent in X if the sequence of partial sums $s_n := \sum_{k=1}^{n} x_k$ is convergent in X.) *Hint:* Go through the proof (b) $\implies$ (c) in the last theorem.

Compare this result with Theorem 6.1.3.

If $J = [a, b]$ is an interval, we let $\ell(J) := b - a$, the length of the interval. We shall repeatedly use the following two trivial observations.

(i) If $x, y \in [a, b]$, then $|x - y| \leq b - a$.

(ii) Let $[a, b] \subset [c, d]$. Then $c \leq a \leq b \leq d$.

Theorem 6.1.12 (Nested Interval Theorem). *Let $J_n := [a_n, b_n]$ be intervals in $\mathbb{R}$ such that $J_{n+1} \subseteq J_n$ for all $n \in \mathbb{N}$. Then $\cap J_n \neq \emptyset$.*

If, furthermore, we assume that $\lim_{n\to\infty} \ell(J_n) \to 0$, then $\cap_n J_n$ contains precisely one point.

---

[1] Recall that $\sum_{n=1}^{\infty} a_n$ is convergent if the sequence of partial sums $s_n := \sum_{k=1}^{\infty} a_k$ is convergent. The limit $s := \lim_n s_n$ is called the sum of the series $\sum_{k=1}^{\infty} a_k$. Now if the series is convergent, then the sequence (s_n) is Cauchy. Also, if $a_k \geq 0$ for all k, then $s := \text{l.u.b.}\ \{s_n\}$ so that $s_n - s_m \leq s - s_m$ for $m \leq n$.

Proof. The strategy is clear. If there exists $c \in \cap_{n=1}^{\infty} J_n$, then $a_n \leq c \leq b_n$ for all n. In particular, c must be an upper bound for $A := \{a_n : n \in \mathbb{N}\}$. Now the other inequality $c \leq b_k$ for all k says that each b_k must be an upper bound for A. This suggests that a choice of c could be the least upper bound of A and we need to show also that each b_k is an upper bound. The reader should draw pictures and devise a proof using the strategy.

For completeness sake, we give the proof.

Note that the hypothesis means that $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$ for all n. In particular, $a_n \leq a_{n+1}$ and $b_{n+1} \leq b_n$ for all $n \in \mathbb{N}$. See Figure 6.1. Observe also that if $s \geq r$, then $J_s \subset J_r$:

$$J_s \subseteq J_{s-1} \subseteq \cdots \subseteq J_{r+1} \subseteq J_r.$$

Hence, in view of the fact (ii) quoted above, we have

$$a_r \leq a_s \leq b_s \leq b_r, \quad \text{in particular } a_s \leq b_r. \tag{6.1}$$

Let E be the set of left endpoints of J_n. Thus, $E := \{a \in \mathbb{R} : a = a_n \text{ for some } n\}$. E is nonempty.

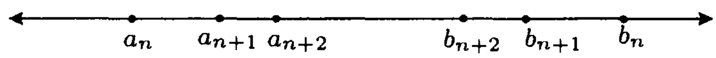

Figure 6.1: Illustration for Nested Interval Theorem

We claim that b_k is an upper bound for E for each $k \in \mathbb{N}$, i.e., $a_n \leq b_k$ for all n and k. If $k \leq n$ then $[a_n, b_n] \subseteq [a_k, b_k]$ and hence $a_n \leq b_n \leq b_k$. (See Figure 6.1.) If $k > n$ then $a_n \leq a_k \leq b_k$. (Use Eq. 6.1.) Thus the claim is proved. By the LUB axiom there exists $c \in \mathbb{R}$ such that $c = \sup E$. We claim that $c \in J_n$ for all n. Since c is an upper bound for E we have $a_n \leq c$ for all n. Since each b_n is an upper bound for E and c is the least upper bound for E we see that $c \leq b_n$. Thus we conclude that $a_n \leq c \leq b_n$ or $c \in J_n$ for all n. Hence $c \in \cap J_n$.

Let us now assume further that $\ell(J_n) \to 0$ as $n \to \infty$. By the first part we know that $\cap J_n \neq \emptyset$. Let $x, y \in \cap_n J_n$. We claim that $x = y$. For, since $x, y \in J_n$ for all n, we have

$$|x - y| \leq \ell(J_n) \to 0, \text{ as } n \to \infty.$$

Thus we conclude that $|x - y| = 0$ and hence $x = y$. $\square$

Remark 6.1.13. Did you observe that the condition $\ell(J_n) \to 0$ is needed only to prove the uniqueness of the common point?

Theorem 6.1.14 (Cantor's Intersection Theorem). *Let (X, d) be a complete metric space. Let a nonempty closed subset F_n be given for all $n \in \mathbb{N}$ such that $F_{n+1} \subset F_n$. Assume further that* $\operatorname{diam}(F_n) \to 0$. *Show that $\cap_{n=1}^{\infty} F_n$ consists exactly of one point.*

Proof. We shall give a sketch. Choose $x_n \in F_n$ for each $n \in \mathbb{N}$. For $n, m \geq N$, $x_m, x_n \in F_n$ so that $d(x_m, x_n) \leq \operatorname{diam}(F_N)$. Hence (x_n) is Cauchy. Since X is complete, (x_n) converges to some point $x \in X$. We claim that $x \in F_n$ for all n. Fix $n \in \mathbb{N}$. Then the sequence $(x_k)_{k \geq n}$ is a sequence in F_n and it converges to x. Since F_n is closed, it follows that $x \in F_n$. Hence $x \cap_{n \in \mathbb{N}} F_n$.

Uniqueness is seen as in the Nested interval theorem. □

Remark 6.1.15. Could you have concluded that $\cap_n F_n \neq \emptyset$ if you did not assume the condition that $\operatorname{diam}(F_n) \to 0$? Compare this with the nested interval theorem. Consider $\mathbb{N}$ with the discrete metric and $F_n := \{k : k \geq n\}$. Then $\cap_n F_n = \emptyset$. See also the next exercise.

Ex. 6.1.16. Consider $\mathbb{N}$ with the metric

$$d(m,n) := \begin{cases} 0 & \text{if } m = n \\ 1 + 1/(m+n) & \text{otherwise.} \end{cases}$$

Show that

(i) Every Cauchy sequence is "eventually" constant and hence $(\mathbb{N}, d)$ is a complete metric space.

(ii) $S_n := \{m \in \mathbb{N} : d(m,n) \leq 1 + 1/2n\} = \{m : m \geq n\}$ is a nested sequence of closed balls whose intersection is empty.

Why does this not contradict Cantor's intersection theorem?

Ex. 6.1.17. Let (X, d) and (Y, d) be complete metric spaces. Show that the product metric space is also complete.

Ex. 6.1.18. Let (X, d) be a metric space such that $d(A, B) > 0$ for any pair of disjoint closed subsets A and B. Show that (X, d) is complete. *Hint:* If (x_n) is Cauchy with distinct terms and which is not convergent, consider $A := \{x_{2k+1} : k \in \mathbb{N}\}$.

Example 6.1.19. We show that $(C[0,1], \| \ \|_1)$ is not complete. Most often a naive guess is to consider the sequence (x^n) in $(C[0,1], \| \ \|_1)$. As seen in Ex. 2.1.13, the sequence does converge to the zero function in $\| \ \|_1$. So what we need is to start with a function which is discontinuous, say, at an interior point and which can be the 'limit' of a sequence of continuous functions. To make the notation simpler, we shall show that $(C[-1,1], \| \ \|_1)$ is not complete.

The strategy is as follows. Construct a continuous function f_n which is 0 on $[-1,0]$, 1 on $[1/n,1]$ and linear on $[0,1/n]$. This is Cauchy and it does not converge to a continuous function in $\| \ \|_1$. This is a subtle step in the proof.

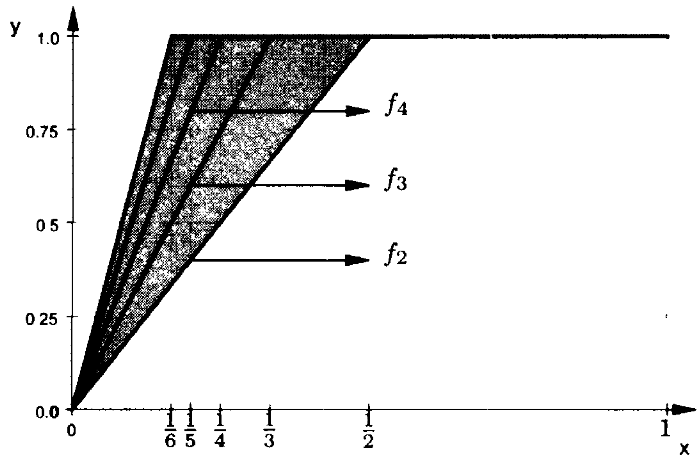

Figure 6.2: Graphs of f_n's of Example 6.1.19

We now give the details. First we write down the function explicitly:

$$f_n(x) :== \begin{cases} 0 & \text{for } x \in [-1,0] \\ nx & \text{for } x \in (0,1/n] \\ 1 & \text{for } x \in [1/n,1] \end{cases}$$

See Figure 6.2. If you look at the picture and recall the geometric meaning of $\| \ \|_1$, then it is the 'shaded area' which goes to 0 as $m,n \to \infty$. We shall give explicit estimate for $\|f_n - f_m\|_1$ to quell your doubts, if any. For $n > m$, we have

$$\begin{aligned} \|f_n - f_m\|_1 &= \int_{-1}^{0} |f_n - f_m| + \int_0^{1/n} |f_n - f_m| \\ &\quad + \int_{1/n}^{1/m} |f_n - f_m| + \int_{1/m}^{1} |f_n - f_m| \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned}$$

Clearly, $I_1 = 0$ and so is I_4. Let us look at I_2:

$$I_2 := \int_0^{1/n} (n-m)x\,dx = \frac{(n-m)}{2n^2} \leq \frac{n}{2n^2} = \frac{1}{2n}.$$

Now let us estimate I_3. First we observe that $f_n(x) = 1$ for $1/n \leq x \leq 1$.

Hence,

$$\begin{aligned} I_3 &:= \int_{1/n}^{1/m} (1 - mx)\, dx \\ &\le \int_{1/n}^{1/m} 1\, dx \qquad (6.2) \\ &\le \frac{1}{m} - \frac{1}{n} \\ &= \frac{n-m}{mn} \\ &\le \frac{n}{mn} = \frac{1}{m}. \qquad (6.3) \end{aligned}$$

So, given $\varepsilon > 0$, if we choose $N > 1/\varepsilon$ and assume that $n > m \ge N$, the inequality (6.3) shows that $\| f_n - f_m \|_1 < \varepsilon$ for $m, n \ge N$. Hence the sequence (f_n) is Cauchy. (We wanted to give precise ε-N argument. In fact, we could have stopped at (6.2), since the sequence $(1/n)$ is convergent and hence is Cauchy! See also the proof of Proposition 2.3.6 (1).)

Now we show that the sequence (f_n) is not convergent in the space $(C[-1,1], \| \ \|_1)$. Let f_n converge to f in the space. We then have

$$\begin{aligned} \| f - f_n \|_1 &= \int_{-1}^{0} |f - f_n| + \int_{0}^{1/n} |f - f_n| + \int_{1/n}^{1} |f - f_n| \\ &= J_1 + J_2 + J_3 \to 0, \end{aligned}$$

as $n \to \infty$. Hence each of the terms (being a sum of nonnegative terms) goes to zero. Let us look at J_1. Since $f_n = 0$ on $[-1, 0]$, we see that $J_1 = \int_{-1}^{0} |f|$. This is independent of n and saying that this goes to zero as $n \to \infty$ is same as saying that this constant is zero. Hence (by an argument we have seen in Example 1.1.10), $|f|$ and hence f is zero on $[-1, 0]$. Next consider J_3. We have $J_3 \to 0$ as $n \to \infty$. We fix N. Then for any $n \ge N$, we have $f_n(x) = 1$ on $[1/N, 1]$. Hence

$$J_3 \ge \int_{1/N}^{1} |f - f_n| = \int_{1/N}^{1} |f - 1|.$$

Since $J_3 \to 0$ as $n \to \infty$, it follows that $\int_{1/N}^{1} |f - 1| = 0$, that is, $f(x) = 0$ for $x \in [1/N, 1]$. Since N is arbitrary, it follows that $f(x) = 0$ for $x \in (0, 1]$. Thus if $f = \lim f_n$ in the NLS, then

$$f(x) = \begin{cases} 0 & \text{if } x \in [-1, 0] \\ 1 & \text{if } x \in (0, 1]. \end{cases}$$

Thus f cannot be continuous. This contradictions shows that the Cauchy sequence (f_n) is not convergent in $(C[-1,1], \| \ \|_1)$. $\square$

Ex. 6.1.20. Let $\mathbf{c}_0$ denote the real vector space of all real sequences that converge to 0. Show that $\|x\| := \sup\{|x_n|\}$ defines a norm. Is $\mathbf{c}_0$ complete with respect to this norm?

Ex. 6.1.21. Let $\mathbf{c}_{00}$ denote the real vector space of all real sequences such that $x_n = 0$ for all n greater than some N (which may depend on x). Show that $\|x\| := \sup\{|x_n|\}$ defines a norm. Is $\mathbf{c}_{00}$ complete with respect to this norm?

Ex. 6.1.22. Let d be defined on $\mathbb{N} \times \mathbb{N}$ as $d(m,n) := \frac{|m-n|}{mn}$. Show that d defines a metric on $\mathbb{N}$ and that the topology induced by d is the discrete topology. Hence conclude that d is equivalent to the standard metric on $\mathbb{N}$ considered as a subset of $\mathbb{R}$. Is $(\mathbb{N}, d)$ complete?

Ex. 6.1.23. Let $\mathbb{R}^{\mathbb{N}}$ denote the set of all real sequences. We consider it as $\prod_{n\in\mathbb{N}} \mathbb{R}$, the cartesian product of countably infinite number of copies of $\mathbb{R}$. We define a metric δ on $\mathbb{R}^{\mathbb{N}}$ as below:

$$d(x,y) := \sup\{\delta(x_n, y_n)/n : n \in \mathbb{N}\},$$

where $\delta(a,b) := \max\{|a-b|, 1\}$ for $a, b \in \mathbb{R}$. Show that $(\mathbb{R}^{\mathbb{N}}, \delta)$ is complete. *Hint:* Convergence in $(\mathbb{R}^{\mathbb{N}}, \delta)$ is the same as coordinatewise convergence as in the finite dimensional $\mathbb{R}^n$.

Ex. 6.1.24. Let X be a metric space such that any closed and bounded subset of X is compact. Prove that X is complete.

Ex. 6.1.25. Show that 'completeness' is not a topological property. That is, show that there exist two equivalent metrics d_1 and $d_2$on a set X such that (X, d_1) is complete while (X, d_2) is not. *Hint:* Exercise 1.2.75 gives such metrics on $(-1, 1)$.

Ex. 6.1.26. Let $\| \ \|_i$, $i = 1, 2$ be two equivalent norms on a vector space V. Show that $(V, \| \ \|_1)$ is complete iff $(V, \| \ \|_2)$ is complete. (Compare this with the previous exercise.)

6.2 Completion of a Metric Space

Definition 6.2.1. Let (X, d) and (Y, d) be metric spaces. We say that a map $f \colon X \to Y$ is an *isometry* if f preserves the metric, that is,

$$d(f(x_1), f(x_2)) = d(x_1, x_2), \text{ for all } x_1, x_2 \in X.$$

Ex. 6.2.2. Show that the map from $\mathbb{C}$ to $\mathbb{R}^2$ given by $z = x+iy \mapsto (x, y)$ is an isometry.

Ex. 6.2.3. Given any NLS, can you think of a family of isometries? *Hint:* Ex. 1.1.19

Ex. 6.2.4. Show that if A is an $n \times n$ real orthogonal matrix, then the map $x \mapsto Ax$ is an isometry of $\mathbb{R}^n$. (Here $\mathbb{R}^n$ is considered as a set of column vectors, i.e., matrices of type $n \times 1$ so that the matrix product Ax makes sense.)

Ex. 6.2.5. Fix a unit vector $u \in \mathbb{R}^n$. Define $R_u(v) = v - 2\langle v, u\rangle u$. Show that R_u is an isometry. (Geometrically, it is the reflection with respect to the 'plane' determined by the equation $\langle x, u\rangle = 0$.) See Figure 6.3.

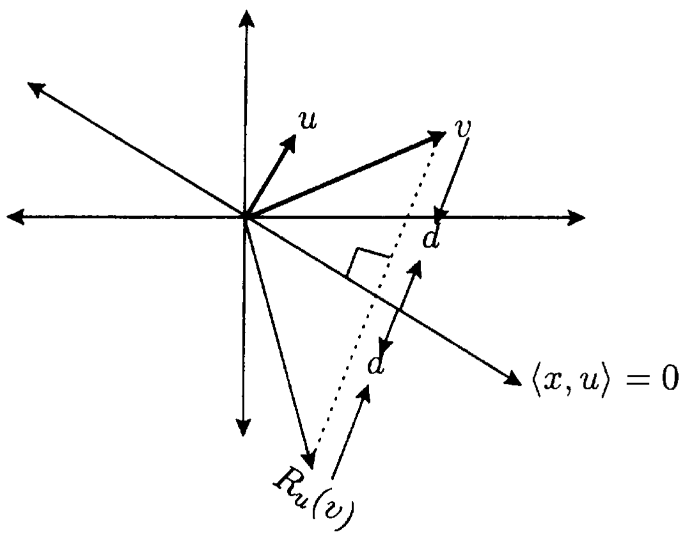

Figure 6.3: Reflection of v

Ex. 6.2.6. The notation is as in Ex. 1.2.73. Show that $\varphi\colon (X, d) \to (Y, \rho)$ is an isometry.

Definition 6.2.7. Let (X, d) be a metric space. A metric space (Y, d) is said to be a *completion* of (X, d) if there exists a map $f\colon X \to Y$ such that (i) f is an isometry of X into Y and (ii) the image $f(X)$ is dense in Y.

Ex. 6.2.8. Show that $\mathbb{R}$ is a completion of $(\mathbb{Q}, |\ |)$.

What is the completion of the space of irrationals with respect to the absolute value metric?

Ex. 6.2.9. Let (X, d) be complete and (Y, d) be a completion of X. What can you conclude?

Ex. 6.2.10. Let V be the vector space of all polynomials with real coefficients endowed with the norm $\|p\| := \sup\{|p(x)| : 0 \le x \le 1\}$. Show that $(C[0,1], \| \ \|_\infty)$ is a completion of V. (Strictly speaking, this is more of a remark than an exercise. See Ex. 2.5.15.)

Theorem 6.2.11 (Completion of a Metric Space). *Let (X, d) be a metric space. Then there exists a completion of (X, d).*

Proof. We shall sketch a modern proof. Let (X, d) be any metric space. Fix a point $o \in X$. For each $x \in X$, consider the function $f_x(y) := d(y, x) - d(y, o)$. Show that $f_x \in B(X)$ and that the map $\varphi \colon x \mapsto f_x$ is an isometry of X into $(B(X), \| \ \|_\infty)$. If we let Y to be closure of $\varphi(X)$ in $B(X)$, then (Y, d_∞) is a completion of (X, d) via the isometry φ of X into Y. (Here d_∞ denotes the restriction of the metric on $B(X)$ to Y.)

We now supply the details, in case you need them. Clearly $f_x \in B(X)$:

$$|f_x(y)| \le |d(x, y) - f(y, o)| \le d(x, o), \text{ for all } y \in X.$$

Hence $\| f_x \|_\infty \le d(x, o)$. To show that the map φ is a isometry, we need to prove that for any $x, y \in X$,

$$d(x, y) = d(\varphi(x), \varphi(y)) = \sup_{z \in X} |f_x(z) - f_y(x)| .$$

We have, for any $z \in X$,

$$\begin{aligned} |f_x(z) - f_y(z)| &= |d(x, z) - d(z, o) - [d(y, z) - d(z, o)]| \\ &= |d(x, z) - d(y, z)| \\ &\le d(x, y). \end{aligned}$$

Also, when $z = y$, we find that $f_x(y) - f_y(y) = d(x, y)$. Thus $\| f_x - f_y \| = d(x, y)$. Thus φ is isometry of X into $B(X)$. If we take Y to be the closure of $\varphi(X)$ in $B(x)$, then Y is complete, being a closed subset of the complete metric space $B(X)$. By our very construction, $\varphi(X)$ is dense in Y. Thus we have a completion of (X, d). □

Ex. 6.2.12. Do you think we can use this method to complete $\mathbb{Q}$?

There is another method due to Cantor to prove the existence of completion of a metric space. While this involves a bit of clumsy notation, the idea has wider ramification and impact. It is typical of mathematics to introduce ideal elements when you can not find them among the ones you have. For instance, you cannot find a real $r \in \mathbb{R}$ such that $r^2 = -1$. So what you do is simply introduce an ideal element i and declare that

$i^2 = -1$ and write expression of the kind $a + ib$ with $a, b \in \mathbb{R}$. You also carry out algebraic operations in the 'usual' way. What you get is the set of complex numbers. (We have not been very precise here, but we hope that you get the idea!)

Before we carry out Cantor's method, we shall prove the following lemma as a preliminary.

Lemma 6.2.13. *Let (X, d) be a metric space.*

(a) *Let (x_n) and (y_n) be Cauchy sequences in X. Then $\lim_{n\to\infty} d(x_n, y_n)$ exists.*

(b) *Let $x_n \to x \in X$ and $d(x_n, y_n) \to 0$. Then $y_n \to x$.*

(c) *Let $d(x_n, x_n') \to 0$ and $d(y_n, y_n') \to 0$. Then*

$$\lim_n d(x_n, y_n) = \lim_n d(x_n', y_n').$$

Proof. The results are given here for easy reference and the proofs are easy and should be attempted by the reader on his own.

To prove (a), it suffices to show that $(d(x_n, y_n))$ is a Cauchy sequences of real numbers. We have

$$\begin{aligned} d(x_n, y_n) - d(x_m, y_m) &\leq d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n) \\ &\quad - d(x_m, y_m) \\ &= d(x_n, x_m) + d(y_m, y_n) \to 0, \end{aligned}$$

as $m, n \to \infty$. (Why?)

Proof of (b) is easy:

$$d(y_n, x) \leq d(y_n, x_n) + d(x_n, x) \to 0.$$

(c) is also easy. Look at

$$d(x_n, y_n) \leq d(x_n, x_n') + d(x_n', y_n') + d(y_n', y_n).$$

Taking limits, we get $\lim_n d(x_n, y_n) \leq \lim_n d(x_n', y_n')$. Similar argument shows the other way inequality and hence the proof. $\square$

Let (X, d) be a metric space. Let (x_n) be a Cauchy sequence which is not convergent in X. We could declare this to be an ideal point to be added to X. But there could be two Cauchy sequences which may converge to the same point in the new space. (For instance, can you think of two distinct sequences in $\mathbb{Q}$ both converging to $\sqrt{2}$? See also (b) of the last lemma.) So what we need to do is to declare that they all represent the same point. We turn this naive idea into a more precise definition.

Definition 6.2.14. We say that two Cauchy sequences (x_n) and (y_n) in a metric space are *equivalent* if $d(x_n, y_n) \to 0$. It is easily seen that this is an equivalence relation $\sim$ on the set of all Cauchy sequences.

Cantor's Construction

Let $\mathfrak{X}$ denote the set of all Cauchy sequences in X. Let $\tilde{X}$ denote the set of equivalence classes in $\mathfrak{X}$ under the equivalence relation $\sim$. We use suggestive Greek letters to denote the elements of $\tilde{X}$. For example, if $[(a_n)]$ is an element of $\tilde{X}$, we denote it by α. If $[(x_n)] \in \tilde{X}$, we denote it by ξ and so on. We also let $\tilde{x}$ stand for the equivalence class of the constant sequence $(x_n := x)$. We have an obvious one-one map φ of X into $\tilde{X}$. It is given by $x \mapsto \tilde{x}$. (Can you prove that this map is one-one?)

We define a metric on $\tilde{X}$, show that it is complete and finally establish that the map $x \mapsto \tilde{x}$ yields a completion of (X, d).

The metric d on $\tilde{X}$ is defined as

$$d(\alpha, \beta) := \lim_n d(a_n, b_n).$$

Part (c) of Lemma 6.2.13 shows that the limit exists and is independent of the choice of the representatives (a_n) and (b_n) of α and β. We prove the triangle inequality:

$$\begin{aligned} d(\alpha, \gamma) &= \lim_n d(a_n, c_n) \\ &\leq \lim_n \left[d(a_n, b_n) + d(b_n, c_n)\right] \\ &= \lim_n d(a_n, b_n) + \lim_n d(b_n, c_n) \\ &= d(\alpha, \beta) + d(\beta, \gamma). \end{aligned}$$

We make an observation: Let $\xi := [(x_n)]$. Then

$$\varphi(x_n) := \tilde{x}_n \to \xi \text{ as } n \to \infty. \tag{6.4}$$

For, if $\varepsilon > 0$ is given, by the Cauchy nature of (x_n), there exists $N \in \mathbb{N}$ such that $d(x_m, x_n) < \varepsilon$ for $m, n \geq N$. We have, for $n \geq N$,

$$d(\tilde{x}_n, \xi) := \lim_{m \to \infty} d(x_n, x_m) < \varepsilon.$$

It is clear that the map $\varphi \colon X \to \tilde{X}$ given by $\varphi(x) = \tilde{x}$ is an isometry.

$$d(\varphi(x), \varphi(y)) := \lim_n d(x, y), \text{ the limit of a constant sequence.}$$

We claim that the image $\varphi(X)$ of X under the map $\varphi \colon x \mapsto \tilde{x}$ is dense in $(\tilde{X}, d)$. Let $\alpha \in \tilde{X}$ and $\varepsilon > 0$ be given. Let (a_n) be a representative of α. Since (a_n) is Cauchy, there exists $N \in \mathbb{N}$ such that $|a_n - a_m| < \varepsilon$ for all $m, n \geq N$. So, if we consider $\tilde{a}_N \in \tilde{X}$, then

$$d(\alpha, \tilde{a}_N) = \lim_n d(a_n, a_N) < \varepsilon.$$

Therefore, $d(\alpha, \varphi(a_N)) < \varepsilon$. This proves that $\varphi(X)$ is dense in $\tilde{X}$.

We next show that $(\tilde{X}, d)$ is complete. Let (α_n) be Cauchy in $\tilde{X}$. Since $\varphi(X)$ is dense in $\tilde{X}$, for each $n \in \mathbb{N}$, we can find an $x_n \in X$ such that $d(\alpha_n, \tilde{x}_n) < 1/n$. We now consider the sequence (x_n) in X. We claim that it is Cauchy in X and that $\alpha_n \to \xi$, where $\xi := [(x_n)]$. Since φ is an isometry, it suffices to show that $(\varphi(x_n))$ is a Cauchy sequence in $\tilde{X}$. Given $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that $d(\alpha_m, \alpha_n) < \varepsilon$ for $m, n \geq N$. We then have for all $m, n \geq N$

$$\begin{aligned} d(\tilde{x}_m, \tilde{x}_n) &\leq d(\tilde{x}_m, \alpha_m) + d(\alpha_m, \alpha_n) + d(\alpha_n, \tilde{x}_n) \\ &< 1/m + \varepsilon + 1/n. \end{aligned}$$

Let $n_0 \geq \max\{1/\varepsilon, N\}$. Then for all $m, n \geq n_0$, the above inequality yields

$$d(\tilde{x}_m, \tilde{x}_n) < 3\varepsilon.$$

Thus (x_n) is Cauchy in X. Let its equivalence class be $\xi \in \tilde{X}$.

We now show that $\alpha_n \to \xi$ as $n \to \infty$.

$$d(\alpha_n, \xi) \leq d(\alpha_n, \tilde{x}_n) + d(\tilde{x}_n, \xi) \leq 1/n + d(\tilde{x}_n, \xi) \to 0,$$

as $d(\tilde{x}_n, \xi) \to 0$ (as $n \to \infty$) by (6.4).

This completes the proof of the fact that $(\tilde{X}, d)$ is a completion of (X, d). □

Remark 6.2.15. Note that Cantor's construction also depends on the completeness of $\mathbb{R}$. (Do you see where we needed it in the above construction?)

The next lemma says that all completions of a given metric spaces are the same in the sense that they are isometric. This is known as the uniqueness of completions.

Lemma 6.2.16 (Uniqueness of Completion). *If (Y, d) and (Z, d) are two completions of the same metric space (X, d), then there is an isometry of Y onto Z.*

Proof. We shall only sketch a proof.

Let $f: X \to Y$ be the map such that f is isometry and $f(X)$ is dense in Y. Let g be the analogous map from X to Z.

Now given any $y \in Y$, there exists a sequence (x_n) in X such that $f(x_n) \to y$. Since $g \circ f^{-1}: f(X) \to g(X)$ is an isometry, it follows that $(g(x_n))$ will be a Cauchy sequence in Z and hence converges to a point, say, $z \in Z$. We define $\varphi(y) = z$. (It is easy to show that $\varphi(y)$ does not depend on the choice of the sequence (x_n) converging to y.) This is the required isometry. □

Ex. 6.2.17. Let (X, d) be a metric space and $D \subset X$ be a dense subset. Assume that every Cauchy sequence in D is convergent in X, that is, if (x_n) is a Cauchy sequence in D then there exists $x \in X$ such that $x_n \to x$. Show that (X, d) is complete. *Hint:* Go through the argument of completeness of $(\tilde{X}, d)$ in the Cantor's construction.

Ex. 6.2.18. Show that a metric space is compact iff every real valued continuous function on X attains a maximum. *Hint:* Completions exist!

Ex. 6.2.19. Assume that (X, d) is not complete. Prove that there exists a uniformly continuous function from $f: X \to (0, \infty)$ such that the $\inf_{x \in X} f(x) = 0$. *Hint:* Completions exist!

6.3 Baire Category Theorem

Definition 6.3.1. A subset $A \subset X$ of a (metric/topological) space is said to be *nowhere dense* in X, if given any nonempty open set U, we can find a nonempty open subset $V \subset U$ such that $A \cap V = \emptyset$.

A prototypical example is a line in $\mathbb{R}^2$. See the next exercise. See also Figure 6.4

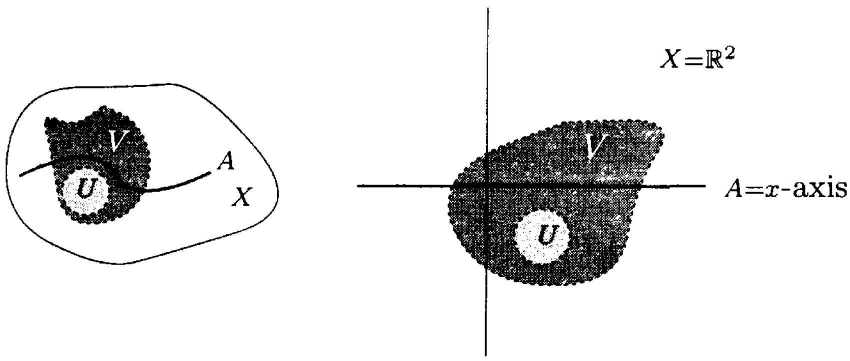

Figure 6.4: Nowhere dense sets

Take sometime to understand the definition. In the past, dense sets are known as everywhere dense sets. Nowhere dense sets are diametrically opposite to dense sets. If we want to say a set is not everywhere dense set, how shall we formulate it? The set may not be dense everywhere but may be dense 'somewhere'. For instance, look at the set $\mathbb{Q} \cap (0,1)$. It is not dense everywhere but dense in any open interval $J \subset (0,1)$. Again, if we want to say a set is nowhere dense in X, how do we formulate it? If you think over these questions, you will gain insight into our definition.

Ex. 6.3.2. Show that the set $\mathbb{Z}$ is nowhere sense in $\mathbb{R}$. How about $\mathbb{Q} \subset \mathbb{R}$?

Show that the x-axis given by $y = 0$ is nowhere dense in $\mathbb{R}^2$.

Ex. 6.3.3. Let V be any proper vector subspace of $\mathbb{R}^n$. Show that V is nowhere dense in $\mathbb{R}^n$.

More generally, let X be a normed linear space. Let V be any proper closed vector subspace of X. Then V is nowhere dense in X. (Compare this with Ex. 1.2.49.)

Ex. 6.3.4. Show that the set $\{1/n : n \in \mathbb{N}\} \cup \{0\}$ is nowhere dense.

Ex. 6.3.5. Show that $A \subset X$ is nowhere dense in X iff the interior of the closure of A is empty, that is, $(\overline{A})^0 = \emptyset$. (This is the standard definition.)

Ex. 6.3.6. Use the definition of nowhere dense sets in Ex. 6.3.5 to solve Ex. 6.3.4.

Theorem 6.3.7 (Baire Category Theorem). *Let (X, d) be a complete metric space.*

(1) *Let U_n be open dense subsets of X, for $n \in \mathbb{N}$. Then $\cap_n U_n$ is dense in X.*

(2) *Let F_n be nonempty closed subsets of X such that $X = \cup_n F_n$. Then at least one of F_n's has nonempty interior. In other words, a complete metric space cannot be a countable union of nowhere dense closed subsets.*

Proof. We first observe that both the statements are equivalent. For, G is open and dense iff its complement $F := X \setminus G$ is closed and nowhere dense. Hence any one of the statements of the theorem follows from the other by taking complements. So, we confine ourselves to proving the first.

Let $U := \cap_n U_n$. We have to prove that U is dense in X. Let $x \in X$ and $r > 0$ be given. We need to show that $B(x,r) \cap U \neq \emptyset$. Since U_1 is dense and $B(x,r)$ is open there exists $x_1 \in B(x,r) \cap U_1$. Since $B(x,r) \cap U_1$ is open, there exists r_1 such that $0 < r_1 < 1/2$ and $B[x_1, r_1] \subset B(x,r) \cap U_1$. We repeat this argument for the open set $B(x_1, r_1)$ and the dense set U_2 to get $x_2 \in B(x_1, r_1) \cap U_2$. Again, we can find r_2 such that $0 < r_2 < 2^{-2}$ and $B[x_2, r_2] \subset B(x_1, r_1) \cap U_2$. See Figure 6.5. Proceeding this way, for each $n \in \mathbb{N}$, we get $x_n \in X$ and an r_n with the properties

$$B[x_n, r_n] \subset B(x_{n-1}, r_{n-1}) \cap U_n \text{ and } 0 < r_n < 2^{-n}.$$

Clearly, the sequence (x_n) is Cauchy: if $m \leq n$,

$$d(x_m, x_n) \leq d(x_n, x_{n-1}) + \cdots + d(x_{m+1}, x_m) \leq \sum_{k=m}^{n} 2^{-k}.$$

Since $\sum_k 2^{-k}$ is convergent, it follows that (x_n) is Cauchy.

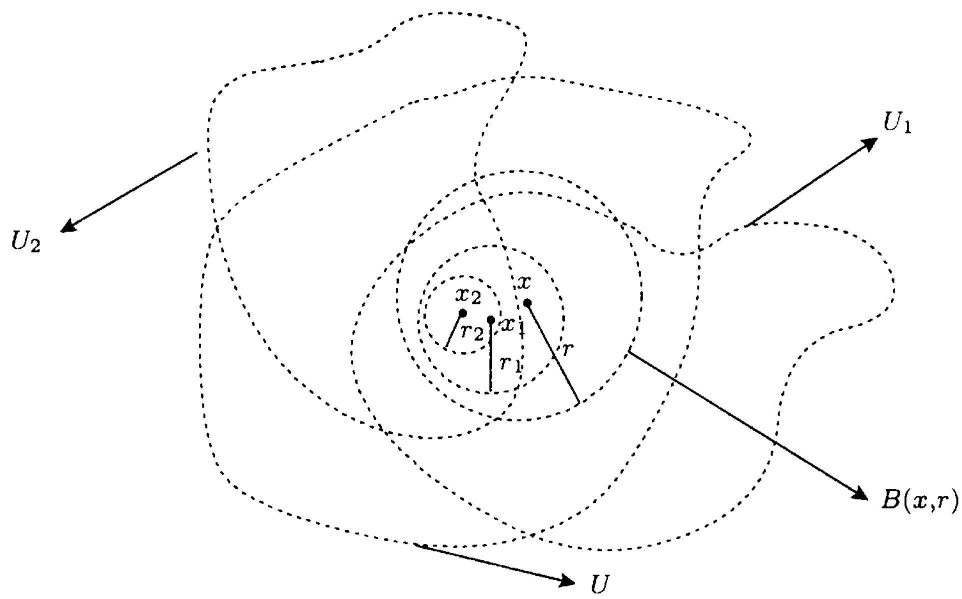

Figure 6.5: Baire Category Theorem

Since X is complete, there exists $x_0 \in X$ such that $x_n \to x_0$. Since x_0 is the limit of the sequence $(x_n)_{n \geq k}$ in the closed set $B[x_k, r_k]$, we deduce that $x_0 \in B[x_k, r_k] \subset B(x_{k-1}, r_{k-1}) \cap U_k$ for all k. In particular, $x_0 \in B(x, r) \cap U$. □

Remark 6.3.8. The importance of our formulation is this. The first statement tells us of a typical way in which Baire category can be used. Imagine that we are on the look-out for an element $x \in X$ with some specific properties. Further assume that the sets of elements which have properties "arbitrarily close" to the desired one are dense open sets in X. Then the result says that there exists at least one element with the desired property. Thus the first formulation is useful when we are interested in the existence problems. This vague way of remembering is well-illustrated especially the proof of the existence of everywhere continuous nowhere differentiable function.

The second formulation says that X cannot be a countable union of "hollow" sets. A typical application: $\mathbb{R}^n$ cannot be the union of a countable collection of lower dimensional subspaces. Another instance: a complete normed linear space cannot be countable dimensional. See Ex. 6.3.13 below.

Ex. 6.3.9. $\mathbb{R}^n$ cannot be the union of a countable collection of lower dimensional subspaces.

Ex. 6.3.10. Does there exist a metric d on $\mathbb{Q}$ which is equivalent to the standard metric but $(\mathbb{Q}, d)$ is complete? *Hint:* Baire!

Ex. 6.3.11. We say that $x \in X$ is an *isolated point* in the metric space (X, d) if there exists an $r > 0$ such that $B(x, r) \cap (X \setminus \{x\}) = \emptyset$.

Let (X, d) be complete. Can the set of isolated points be countably infinite?

Ex. 6.3.12. Show that any countable complete metric space has an isolated point.

Ex. 6.3.13. Let X be an infinite dimensional complete normed linear space. Show that X cannot be countable dimensional.

Ex. 6.3.14. Use Baire's theorem to show that $\mathbb{R}$ is uncountable.

Ex. 6.3.15 (Uniform Boundedness Principle). Let X be a complete metric space and $\mathcal{F}$ a family of continuous real valued functions on X. Assume that for all $x \in X$, there exists $C_x > 0$ such that $|f(x)| \leq C_x$ for all $f \in \mathcal{F}$. Then there exists a non-empty open set $U \subseteq X$ and a constant C such that $|f(x)| \leq C$ for all $f \in \mathcal{F}$ and $x \in U$.

Everywhere Continuous and Nowhere Differentiable Functions

Definition 6.3.16 (Saw-tooth functions). A function $f\colon [0,1] \to \mathbb{R}$ is called a *saw-tooth* function if there exist a positive real number h and a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ of the interval $[0,1]$ such that $f(t_i) = 0$ for $0 \leq i \leq n$ and $f(s_i) = h$ where s_i is the midpoint of the interval $[t_{i-1}, t_i]$ and extended linearly to all of $[0,1]$. See Figure 6.6. We shall refer to h as the height of the teeth.

Theorem 6.3.17. *In the space $(C[0,1], \| \ \|_\infty)$, the subset $\mathcal{A}$ of functions that are not differentiable at any point of $[0,1)$ is dense.*

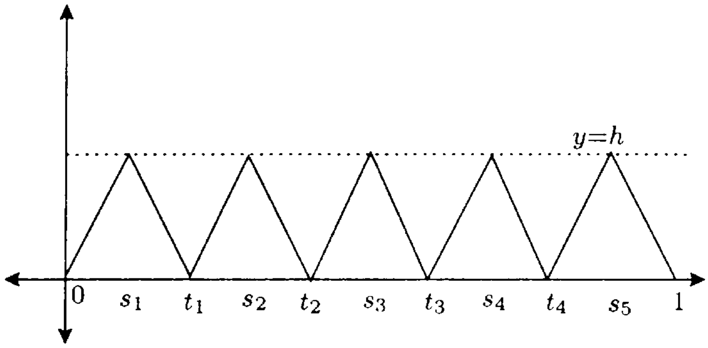

Figure 6.6: Saw-tooth function

Proof. This is perhaps the most difficult proof in the entire book. We suggest that the reader goes through the proof first to get some idea of the proof, later to look carefully at the details and last to think over the main steps. The basic strategy is spelled out in the next couple of paragraphs. Unlike the other proofs in the book, the strategy alone will not suffice to complete the proof on our own. Lots of new ideas are involved and we get a taste of real top-class mathematics.

If $f \in C[0,1]$ has a derivative at $x \in [0,1)$, then the right-hand difference quotients

$$\frac{f(x+h)-f(x)}{h}, \qquad (0 < h < 1-x)$$

are bounded. Thus, for large values of $n \in \mathbb{N}$, the function f belongs to the complement of the set $\mathcal{A}_n$ consisting of those functions $f \in C[0,1]$ such that for each $x \in [0, 1-1/n]$, we can find h with $0 < h < 1-x$ such that $|f(x+h) - f(x)| > nh$. (Take sometime to digest this.)

Clearly, $\cap_{n=1}^{\infty} \mathcal{A}_n \subset \mathcal{A}$. If we show that the smaller set $A \cap_n \mathcal{A}_n$ is dense, we are through. In view of Baire category theorem (Remark 6.3.8), it suffices to show that each $\mathcal{A}_n$ is an open dense set. For the remainder of the proof, we fix n.

We show that $\mathcal{C}_n := C[0,1] \setminus \mathcal{A}_n$ is closed in $C[0,1]$. It suffices to show that if $\{f_k\} \subseteq \mathcal{C}_n$, with $\lim f_k = f \in \mathcal{C}[0,1]$, then $f \in \mathcal{C}_n$. Now, $f_k \in \mathcal{C}_n$ implies that there exists $t_k \in [0,1]$ such that

$$\left|\frac{f_k(t_k+h) - f_k(t_k)}{h}\right| \le n, \quad \text{for all } h.$$

Since $[0,1]$ is compact, the sequence t_k has a convergent subsequence.

We call this subsequence again by t_k and let $t_0 = \lim t_k$. Then

$$\begin{aligned}
\left|\frac{f(t_0+h)-f(t_0)}{h}\right| &\le \left|\frac{f(t_0+h)-f(t_k+h)}{h}\right| \\
&\quad + \left|\frac{f(t_k+h)-f_k(t_k+h)}{h}\right| \\
&\quad + \left|\frac{f_k(t_k+h)-f_k(t_k)}{h}\right| \\
&\quad + \left|\frac{f_k(t_k)-f(t_k)}{h}\right| \\
&\quad + \left|\frac{f(t_k)-f(t_0)}{h}\right| \\
&= (1)+(2)+(3)+(4)+(5), \text{ say.}
\end{aligned}$$

Now, fix h. For any $\varepsilon > 0$, if k is large enough, (1) and (5) are smaller than ε, since f is continuous, and $t_k \to t_0$. (2) and (4) are smaller than ε, since f_k converges uniformly to f. The third term (3) is $\le n$. Hence, we get

$$\left|\frac{f(t_0+h)-f(t_0)}{h}\right| \le n + 4\varepsilon \quad \text{for any } \varepsilon > 0$$

so that

$$\left|\frac{f(t_0+h)-f(t_0)}{h}\right| \le n$$

and hence $f \in \mathcal{C}_n$. Thus each $\mathcal{C}_n$ is closed.

We now show that $\mathcal{A}_n$ is dense in $C[0,1]$. Let $f \in C[0,1]$ and $\varepsilon > 0$ be given. By Proposition 4.2.13 (page 94), there exists a piecewise linear $p \in C[0,1]$ such that $\|f-p\|_\infty < \varepsilon/2$. Let the slopes of the line segment comprising of the graph of p be $m_1, \ldots, m_k$. Choose an integer $m > n + \max\{m_i : 1 \le i \le k\}$. Let s be a saw-tooth piecewise linear functions whose line segments have slopes $\pm m$ and for which $0 \le s(x) \le \varepsilon/2$ for $x \in [0,1]$. We set $g := p + s$. Then $g \in C[0,1]$ and $\|p-g\|_\infty = \max\{s(x) : x \in [0,1]\} = \varepsilon/2$.

We claim that $g \in \mathcal{A}_n$. Let $0 < x < 1 - 1/n$. Choose $0 < h < 1 - x$ so small that the points $(x, p(x))$ and $(x+h, p(x+h))$ both lie on a line segment of the graph of p, of slope m_i, and at the same time the points $(x, s(x))$ and $(x+h, s(x+h))$ both lie on a line segment of the graph of s. (Why is all this possible? See Figure 6.7.) Then,

$$\begin{aligned}
|p(x+h)-p(x)| &= |m_i|\,h \\
|s(x+h)-s(x)| &= mh > nh + |m_i|\,h.
\end{aligned}$$

Hence $|g(x+h)-g(x)| > nh$, that is, $g \in A_n$. □

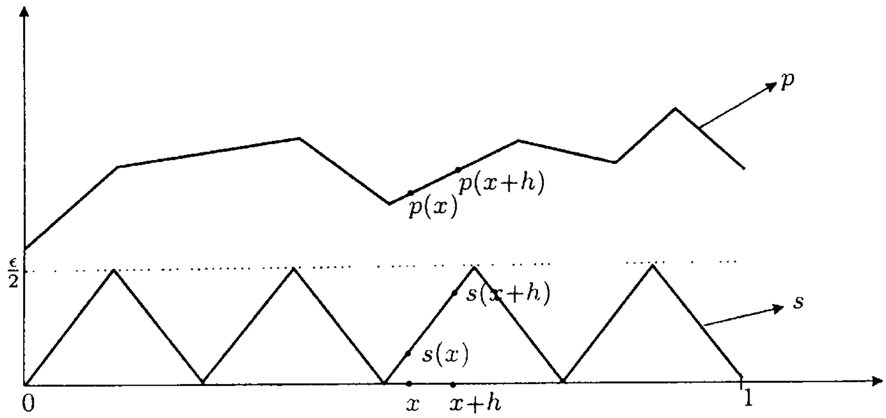

Figure 6.7: Graph of P and s of Theorem 6.2

6.4 Banach's Contraction Principle

Definition 6.4.1. Let X and Y be metric spaces. A map $T\colon X \to Y$ is said to be a *contraction* if there exists a constant c, $0 < c < 1$ such that

$$d(T(x), T(x')) \leq c d(x, x'), \qquad \text{for all } x, x' \in X.$$

Ex. 6.4.2. Show that any contraction is Lipschitz continuous and hence it is uniformly continuous.

Ex. 6.4.3. Let $f\colon [a, b] \to [a, b]$ be differentiable and $|f'(x)| \leq c$ with $0 < c < 1$. Then f is a contraction of $[a, b]$.

Ex. 6.4.4. Let X and Y be metric spaces. Assume that Y is a discrete metric space and that $f\colon X \to Y$ is a contraction. What can you conclude about f?

Theorem 6.4.5 (Banach Contraction Principle). *Let (X, d) be a complete metric space. Assume that $T\colon X \to X$ is a contraction. Then f has a unique fixed point, that is, a point $x \in X$ such that $f(x) = x$.*

Proof. The strategy of the proof is this. We take any $x_0 \in X$ and define a sequence recursively by setting $x_1 = Tx_0$ and $x_{n+1} = Tx_n$ for $n \geq 2$. We show that (x_n) is a Cauchy sequence so that (x_n) converges to an $x \in X$. It is easy to show that $Tx = x$ and that such an x is unique.

Let us work out the details. Let $x_0 \in X$ be an arbitrary point. If $Tx_0 = x_0$, then we got what we wanted. So, we assume that $Tx_0 \neq x_0$. We defined recursively a sequence by setting

$$x_1 := Tx_0,\ x_2 := Tx_1,\ \ldots,\ x_{n+1} = Tx_n \quad \text{for } n \in \mathbb{N}.$$

We claim that the sequence (x_n) is Cauchy in X. First of all, let us observe the following: For any $n \geq 3$,

$$\begin{aligned} d(x_n, x_{n-1}) &= d(Tx_{n-1}, Tx_{n-2}) \\ &\leq c \cdot d(x_{n-1}, x_{n-2}) \\ &= c \cdot d(Tx_{n-2}, Tx_{n-3}) \\ &\leq c^2 \cdot d(x_{n-2}, x_{n-3}) \\ &\vdots \\ &\leq c^{n-1} \cdot d(x_1, x_0). \end{aligned} \tag{6.5}$$

Hence, by triangle inequality, we have, for $m < n$

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \cdots + d(x_{n-1}, x_n) \\ &\leq \left(c^m + c^{m+1} + \cdots + c^{n-1}\right) d(x_1, x_0) \\ &= c^m d(x_0, x_1) \left(1 + c + \cdots + c^{n-m-1}\right) \\ &\leq c^m \frac{d(x_0, x_1)}{1-c}. \end{aligned} \tag{6.6}$$

It follows from (6.6) that (x_n) is Cauchy. Let $x := \lim x_n$. Using continuity of T we see that $x = \lim_n T(T^n x) = Tx$.

Uniqueness is easy. Let $x, y \in X$ be such that and $Tx = x$ and $Ty = y$. Then $d(x, y) = d(Tx, Ty) \leq cd(x, y)$. If $d(x, y) \neq 0$, we arrive at the contradiction $d(x, y) < d(x, y)$. □

Remark 6.4.6. Note the the proof above shows that x_n's are arbitrarily close to the fixed point x and indeed we have

$$d(T^n x_0, x) \leq \frac{c^n}{1-c} d(x_0, Tx_0).$$

Ex. 6.4.7. Let (X, d) be a complete metric space. Assume that $f \colon X \to X$ is a map such that for some positive integer the k-times composition $f \circ \cdots \circ f$ (composition of f with itself k times) is a contraction. Prove that f has unique fixed point.

Ex. 6.4.8. Let a, b be real numbers with $0 < b < 1$. Consider the subset $X \subset C[0, b]$ consisting of functions f such that $f(0) = a$. Then X is closed in $C[0, b]$. Define

$$Tf(x) := a + \int_0^x |f(t)|\, dt, (0 \leq x \leq b).$$

Prove that T is a contraction of x and hence there exists a unique $f \in X$ which satisfies $f' = |f|$ on $(0, b)$.

Ex. 6.4.9. Show that the map T on $(C[0,1], \| \|_\infty)$ defined by

$$Tf(x) := \int_0^x (x-t)f(t)\,dt, 0 \le x \le 1, f \in C[0,1]$$

is a contraction. What is its fixed point?

Remark 6.4.10. There are two standard applications of the contraction principle in analysis. One is the existence of solutions of an ordinary differential equation. The other is to prove the inverse or implicit function theorem in calculus of several variables. Versions of these will be given below. For more details, we refer the reader to literature.

Implicit Function Theorem

First we give an easy version of the implicit function theorem. The version one needs in calculus of two variables is given next.

Theorem 6.4.11 (An Implicit Function Theorem). *Let D be the rectangle $D := \{(x,y) \in \mathbb{R}^2 : a \le x \le b$ and $c \le y \le d\}$. Assume that $f : D \to \mathbb{R}$ has continuous partial derivatives. Then there exists a unique function $g : [a,b] \to \mathbb{R}$ such that $f(x, g(x)) = 0$ for all $x \in [a,b]$.*

Proof. We shall sketch the argument. Let m, M be such that

$$m \le \frac{\partial f}{\partial y}(x,y) \le M, \text{ for } (x,y) \in D.$$

Let $X := (C[a,b], \| \|_\infty)$. Consider

$$T(\varphi)(x) := \varphi(x) - \frac{1}{M} f(x, \varphi(x)), \text{ for } x \in [a,b].$$

One shows that T is a contraction by an obvious use of the mean value theorem. □

Theorem 6.4.12 (Implicit Function Theorem). *Let $U \subset \mathbb{R}^2$ be open and $f : U \to \mathbb{R}$ have continuous partial derivatives, f_x and f_y on U. Assume that $(x_0, y_0) \in U$ is such that $f(x_0, y_0) = 0$ and that $\frac{\partial f}{\partial y}(x_0, y_0) \ne 0$. Then there exist an $\varepsilon > 0$ and a function $g : (x_0 - \varepsilon, x_0 + \varepsilon) \to \mathbb{R}$ such that the following hold:*

(1) *$f(x, g(x)) = 0$ for all $x \in (x_0 - \varepsilon, x_0 + \varepsilon)$*

(2) *g is differentiable on its domain with*

$$g'(x) = -\frac{\frac{\partial f}{\partial x}(x, g(x))}{\frac{\partial f}{\partial y}(x, g(x))}.$$

Proof. Let

$$F(x,y) := y - \left(\frac{\partial f}{\partial y}(x_0,y_0)\right)^{-1} f(x,y).$$

Then $F(x_0,y_0) = y_0$ and $\frac{\partial F}{\partial y}(x_0,y_0) = 0$. Since $\frac{\partial F}{\partial y}$ is continuous at (x_0,y_0), we infer that $\frac{\partial F}{\partial y}(x,y)$ is small for (x,y) for (x,y) near (x_0,y_0).

We choose $\delta_1 > 0$ and $\delta_2 > 0$ so that whenever $|x-x_0| < \delta_1$ and $|y-y_0| < \delta_2$, then

$$\left|\frac{\partial F}{\partial y}(x,y)\right| \leq 1/2.$$

We may assume, by taking smaller δ_1, that $|F(x,y_0) - y_0| \leq \delta_2/2$ for all x with $|x-x_0| < \delta_1$.

Let X be the set of continuous functions g on $[x_0-\delta_1, x_0+\delta_1]$ such that $g(x_0) = y_0$ and $|g(x) - y_0| \leq \delta_2$ if $|x-x_0| \leq \delta_1$. We let $d(g_1,g_2) := \sup\{|g_1(x) - g_2(x)| : |x-x_0| \leq \delta_1\}$. Then it is easy to see that (X,d) is a complete metric space. (This is a good exercise. Do it now!)

For $g \in X$, we define $T(g)(x) := F(x,g(x))$. Then $T(g)(x_0) = y_0$ and

$$\begin{aligned}
|(Tg)(x) - y_0| &= |F(x,g(x)) - y_0| \\
&\leq |F(x,g(x)) - F(x,y_0)| + |F(x,y_0) - y_0| \\
&\leq \left|\frac{\partial F}{\partial y}(x,\eta)\right| |g(x) - y_0| + \delta_2/2 \\
&\leq \frac{\delta_2}{2} + \frac{\delta_2}{2} = \delta_2.
\end{aligned}$$

Thus, T maps X to itself. We claim that T is a contraction:

$$\begin{aligned}
|(Tg)(x) - (Th)(x)| &= |F(x,g(x)) - F(x,h(x))| \\
&= \left|\frac{\partial F}{\partial y}(x,\eta)\right| |g(x) - h(x)| \\
&\leq \frac{1}{2}\,|g(x) - h(x)|\,.
\end{aligned}$$

Let $g_0(x) = y_0$ for $x \in J := [x_0-\delta_1, x_0+\delta_1]$. Define $g_{n+1} := Tg_n$. Then (g_n) is uniformly Cauchy on J and hence converges to a continuous function g. We also have

$$F(x,g(x)) = \lim_{n\to\infty} F(x,g_n(x)) = \lim_{n\to\infty} g_{n+1}(x) = g(x).$$

Therefore, g is a fixed point of T. Consequently, $F(x,g(x)) = g(x)$ and hence $f(x,g(x)) = 0$.

We now show that g is differentiable and compute its derivative. Assume that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are continuous at $(x,g(x))$. Let x_1 be such that

$|x_1 - x_0| < \delta_1$. Then $f(x, g(x)) = 0$ and $f(x_1, g(x_1)) = 0$. We have

$$\begin{aligned} 0 &= f(x_1, g(x_1)) - f(x, g(x)) \\ &= f(x_1, g(x_1)) - f(x_1, g(x)) + f(x_1, g(x)) - f(x, g(x)) \\ &= \frac{\partial f}{\partial y}(x_1, \eta)[g(x_1) - g(x)] + \frac{\partial f}{\partial x}(\xi, g(x))[x_1 - x], \end{aligned}$$

for some ξ between x and x_1 and for some η between $g(x)$ and $g(x_1)$. Therefore,

$$\frac{g(x_1) - g(x)}{x_1 - x} = -\frac{\partial f}{\partial x}(\xi, g(x)) \left(\frac{\partial f}{\partial y}(x_1, \eta) \right)^{-1}.$$

Hence

$$g'(x) = \lim_{x_1 \to x} \frac{g(x_1) - g(x)}{x_1 - x} = -\frac{\partial f}{\partial x}(x, g(x)) \left(\frac{\partial f}{\partial y}(x, g(x)) \right)^{-1},$$

by the continuity of the partial derivatives. □

Theorem 6.4.13 (Picard's Existence Theorem). *Consider the domain* $D := \{(x, y) \in \mathbb{R}^2 : |x - x_0| < a, |y - y_0| < b\}$. *Assume that* $f\colon D \to \mathbb{R}$ *is a continuous function satisfying the Lipschitz condition in the* y*-variable uniformly in the* x*-variable, that is,*

$$|f(x, y_1) - f(x, y_2)| \le L\,|y_1 - y_2|, \qquad (x, y_1), (x, y_2) \in D.$$

Then there exists a $\delta > 0$ *and* $g\colon [x_0 - \delta, x_0 + \delta] \to \mathbb{R}$ *which is a solution of the initial value problem:*

$$g'(x) = f(x, g(x)) \text{ satisfying the initial condition } g(x_0) = y_0. \tag{6.7}$$

Proof. We shall only give a broad outline of the proof and leave the details for the reader to work out.

Observe that the given initial value problem (6.7) is, by an application of the fundamental theorem of calculus, equivalent to the integral equation

$$g(x) = y_0 + \int_{x_0}^{x} f(t, g(t))\,dt, \qquad (|x - x_0| < \delta). \tag{6.8}$$

Let $M > 0$ be such that $|f(x, y)| \le M$ for $(x, y) \in D$. Choose $\delta > 0$ such that

$$L\delta < 1 \text{ and } [x_0 - \delta, x_0 + \delta] \times [y_0 - M\delta, y_0 + M\delta] \subset D.$$

Define

$$Y := \{f \in C[x_0 - \delta, x_0 + \delta] : |g(x) - y_0| \leq M\delta \text{ for } |x - x_0| \leq \delta\}.$$

Then Y is closed in $(C[x_0 - \delta, x_0 + \delta], \| \ \|_\infty)$ and hence is a complete metric space. The map

$$Tg(x) := y_0 + \int_{x_0}^{x} f(t, g(t))\, dt$$

is a contraction on Y. □

Ex. 6.4.14. Let $a > 0$ and $g \in C[0, a]$. Define $T\colon C[0, a] \to C[0, a]$ as follows:

$$Tf(x) := \int_0^x f(t)\, dt + g(x), f \in C[0, a], 0 \leq x \leq a.$$

Show that T is a contraction on $(C[0, a], \| \ \|_\infty)$ iff $a < 1$.

Assume that g is differentiable. Find the initial value problem of an ordinary differential equation whose solution is the fixed point of T.

Bibliography

The first two books are recommended for futher study of point-set topology. The next two are referred to in the book. Interested readers may also look at some of my articles on Topology in the website mentioned below.

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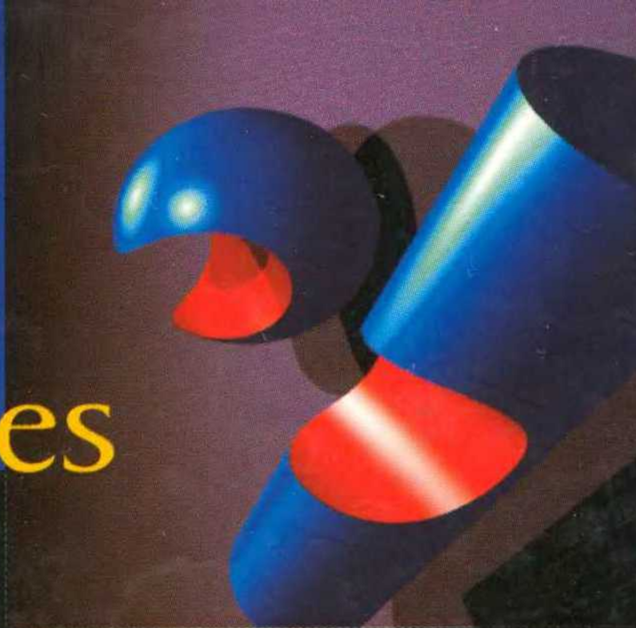
Topology of Metric Spaces



S. Kumaresan

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